

Integral theory for Hopf Algebroids

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Abstract

The theory of integrals is used to analyse the structure of Hopf algebroids [1, 6]. We prove that the total algebra of a Hopf algebroid is a separable extension of the base algebra if and only if it is a semi-simple extension and if and only if the Hopf algebroid possesses a normalized integral. The total algebra of a finitely generated and projective Hopf algebroid is a Frobenius extension of the base algebra if and only if the Hopf algebroid possesses a non-degenerate integral. We give also a sufficient and necessary condition in terms of integrals, under which it is a quasi-Frobenius extension, and illustrate by an example that this condition does not hold true in general. Our results are generalizations of classical results on Hopf algebras [20, 28].

1 Introduction

The notion of *integrals* in Hopf algebras has been introduced by Sweedler [34]. The integrals in Hopf algebras over principal ideal domains were analysed in [20, 33] where the following – by now classical – results have been proven:

- A free, finite dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if it possesses a non-degenerate left integral. (Larson-Sweedler Theorem.)
- The antipode of a free, finite dimensional Hopf algebra over a principal ideal domain is bijective.
- A Hopf algebra over a field is finite dimensional if and only if it possesses a non-zero left integral.
- The left integrals in a finite dimensional Hopf algebra over a field form a one dimensional subspace.
- A Hopf algebra over a field is semi-simple if and only if it possesses a normalized left integral. (Maschke Theorem.)

There are numerous generalizations of these results in the literature. Historically the first is due to Pareigis [28] who proved the following statements on a finitely generated and projective Hopf algebra (H, Δ, ϵ, S) over a commutative ring k :

- H is a Frobenius extension of k if and only if there exists a Frobenius functional $\psi : H \rightarrow k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(_)$.
- The antipode, S , is bijective.
- The left integrals form a projective rank 1 direct summand of the k -module H .
- H is a quasi-Frobenius extension of k .
- A finitely generated and projective bialgebra over a commutative ring k , such that $\text{pic}(k) = 0$, is a Hopf algebra if and only if it possesses a non-degenerate left integral.

The generalization of the Maschke theorem to Hopf algebras H over commutative rings k states that the existence of a normalized left integral in H is equivalent to the separability of H over k , what is further equivalent to its relative semi-simplicity in the sense [16, 17] that any H -module is (H, k) -projective [13, 21]. This is equivalent to the true semi-simplicity of H (i.e. the true projectivity of any H -module [29]) if and only if k is a semi-simple ring [21].

As a nice review on these results we recommend Section 3.2 in [14].

Similar results are known also for generalizations of Hopf algebras. Integrals for finite dimensional quasi-Hopf algebras [15] over fields were studied in [18, 26, 27, 12] and for finite dimensional weak Hopf algebras [5, 4] over fields in [4, 42].

The purpose of the present paper is to investigate which of the above results generalizes to *Hopf algebroids*.

Hopf algebroids with bijective antipode have been introduced in [6, 1]. It is important to emphasize that this notion of a Hopf algebroid is not equivalent to that introduced under the same name by Lu in [22]. Here we generalize the definition of [6, 1] by relaxing the requirement of the bijectivity of the antipode. A Hopf algebroid consists of a compatible pair of a left and a right bialgebroid structure [39, 22, 35, 36] on the common total algebra A . The antipode relates these two left- and righthanded structures. Left/right integrals *in* a Hopf algebroid are defined as the invariants of the left/right regular A -module in terms of the counit of the left/right bialgebroid. Integrals *on* a Hopf algebroid are the comodule maps from the total algebra to the base algebra (reproducing the integrals *in* the dual bialgebroids, provided the duals possess bialgebroid structures).

The total algebra of a bialgebroid can be looked at as an extension of the base algebra or its opposite via the source and target maps, respectively. In this way there are four algebra extensions associated to a Hopf algebroid. The main results of the paper relate properties of these extensions to the existence of integrals with special properties:

- A Maschke type theorem, proving that the separability, and also the (in two cases left in two cases right) semi-simplicity of any of the four extensions is equivalent to the existence of a normalized integral *in* the Hopf algebroid (Theorem 3.1).
- The total algebra is a Frobenius extension both of the base algebra and of its opposite algebra if and only if there exists a non-degenerate left (equivalently, right) integral *in* the Hopf algebroid (Corollary 4.8). In particular, if the total algebra is a finitely generated and projective module of the base algebra (in several appropriate senses), then any of the four extensions is a Frobenius extension if and only if there exists a non-degenerate (left or right) integral *in* the Hopf algebroid (Theorem 4.7).
- Under the same finitely generated projectivity conditions in the previous item, any of the four extensions is (in two cases a left in two cases a right) quasi-Frobenius extension if and only if the (left or right) integrals *on* the Hopf algebroid form a flat module over the base algebra (Theorem 5.2).

Our main tool in proving the latter two points is the Fundamental Theorem for Hopf modules over Hopf algebroids (Theorem 4.2).¹

The paper is organized as follows: We start Section 2 with reviewing some results on bialgebroids from [39, 22, 35, 36, 31, 10, 19, 32, 37], the knowledge of which is needed for the understanding of the paper. Then we present the definition of Hopf algebroids and discuss some of its immediate consequences. Integrals both *in* and *on* Hopf algebroids are introduced and some equivalent characterizations are given.

In Section 3 we prove two Maschke type theorems. The first one collects some equivalent properties (in particular separability) of the inclusion of the base algebra in the total algebra of a

¹ In contrast to the Fundamental Theorem for Hopf modules over Hopf algebras, Theorem 4.2 is *not* proven for *arbitrary* Hopf algebroids. A weaker version of the theorem, relying on a more restrictive notion of a comodule of a Hopf algebroid, is proven for an arbitrary Hopf algebroid in Theorem 3.26 and Remark 3.27 of the arXiv version of [3].

Hopf algebroid. These equivalent properties are related to the existence of a normalized integral *in* the Hopf algebroid. The second theorem collects some equivalent properties (in particular coseparability) of the coring underlying a Hopf algebroid. These equivalent properties are shown to be equivalent to the existence of a normalized integral *on* the Hopf algebroid.

In Section 4 we prove the Fundamental Theorem for Hopf modules over a Hopf algebroid. This theorem is somewhat stronger than the one that can be obtained by the application of ([8], Theorem 5.6) to the present situation. Still, it is not known to hold for an arbitrary Hopf algebroid. The main result of the section is Theorem 4.7. In proving it we follow an analogous line of reasoning as in [20]. That is, assuming that all the four module structures of the total algebra over the base algebra are finitely generated and projective, we apply the Fundamental Theorem to the Hopf module, constructed on the dual of the Hopf algebroid (w.r.t. the base algebra). Similarly to the case of Hopf algebras, our result implies the existence of non-zero integrals *on* any finitely generated projective Hopf algebroid. Since the dual of a (finitely generated projective) Hopf algebroid is not known to be a Hopf algebroid in general, we have no dual result, that is, we do not know whether there exist non-zero integrals *in* any finitely generated projective Hopf algebroid. As a byproduct, also a sufficient and necessary condition on a finitely generated projective Hopf algebroid is obtained, under which the antipode is bijective. We do not know, however, whether this condition follows from the axioms.

In Section 5 we use the results of Section 4 to obtain conditions which are equivalent to the (either left or right) quasi-Frobenius property of any of the four extensions behind a finitely generated and projective Hopf algebroid. In order to show that these conditions do not hold true in general, we construct a counterexample.

Throughout the paper we work over a commutative ring k . That is, the total and base algebras of our Hopf algebroids are k -algebras. For an (always associative and unital) k -algebra $A \equiv (A, m_A, 1_A)$ we denote by ${}_A\mathcal{M}$, \mathcal{M}_A and ${}_A\mathcal{M}_A$ the categories of left, right, and bimodules over A , respectively. For the k -module of morphisms in ${}_A\mathcal{M}$, \mathcal{M}_A and ${}_A\mathcal{M}_A$ we write ${}_A\text{Hom}(\ , \)$, $\text{Hom}_A(\ , \)$ and ${}_A\text{Hom}_A(\ , \)$, respectively.

Acknowledgment. I am grateful to the referee for a careful study of the paper and for pointing out an error in the earlier version. His or her constructive comments lead to a significant improvement of the paper.

This work was supported by the Hungarian Scientific Research Fund OTKA – T 034 512, T 043 159, FKFP – 0043/2001 and the Bolyai János Fellowship.

2 Integrals for Hopf algebroids

Hopf algebroids with bijective antipodes have been introduced in [6], where several equivalent reformulations of the definition ([6], Definition 4.1) have been given. The definition we give in this section generalizes the form in ([6], Proposition 4.2 *iii*) by allowing the antipode not to be bijective.

Integrals *in* Hopf algebroids also have been introduced in [6]. As we shall see, the definition ([6], Definition 5.1) applies also in our more general setting. In this section we introduce integrals also *on* Hopf algebroids.

In order for the paper to be self-contained, we recall some results on bialgebroids from [39, 22, 35, 36, 19]. For more on bialgebroids we refer to the papers [31, 10, 32, 37].

The notions of Takeuchi's \times_R -bialgebra [39], Lu's bialgebroid [22] and Xu's bialgebroid with anchor [41] have been shown to be equivalent in [10]. We use the definition in the following form:

Definition 2.1. A *left bialgebroid* $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ consists of two algebras A and B over the commutative ring k , which are called the total and base algebras, respectively. A is a $B \otimes_k B^{op}$ -ring (i.e. a monoid in ${}_{B \otimes B^{op}}\mathcal{M}_{B \otimes B^{op}}$) via the algebra homomorphisms $s : B \rightarrow A$ and $t : B^{op} \rightarrow A$, called the source and target maps, respectively. (This means that the ranges of s and t are commuting subalgebras in A .) In terms of s and t , one equips A with a B - B bimodule structure ${}_B A_B$ as

$$b \cdot a \cdot b' := s(b)t(b')a \quad \text{for } a \in A, b, b' \in B.$$

The triple $({}_B A_B, \gamma, \pi)$ is a B -coring, that is a comonoid in ${}_B \mathcal{M}_B$. Introducing Sweedler's convention $\gamma(a) = a_{(1)} \otimes_B a_{(2)}$ for $a \in A$ (where implicit summation is understood), the axioms

$$a_{(1)} t(b) \otimes_B a_{(2)} = a_{(1)} \otimes_B a_{(2)} s(b) \quad (2.1)$$

$$\gamma(1_A) = 1_A \otimes_B 1_A \quad (2.2)$$

$$\gamma(aa') = \gamma(a)\gamma(a') \quad (2.3)$$

$$\pi(1_A) = 1_B \quad (2.4)$$

$$\pi(a \circ s \circ \pi(a')) = \pi(aa') \quad (2.5)$$

$$\pi(a \circ t \circ \pi(a')) = \pi(aa') \quad (2.6)$$

are required for all $b \in B$ and $a, a' \in A$.

Notice that – although $A \otimes_B A$ is not an algebra – axiom (2.3) makes sense in view of (2.1).

Homomorphisms of left bialgebroids $\mathcal{A}_L = (A, B, s, t, \gamma, \pi) \rightarrow \mathcal{A}'_L = (A', B', s', t', \gamma', \pi')$ are pairs of k -algebra homomorphisms $(\Phi : A \rightarrow A', \phi : B \rightarrow B')$ satisfying

$$s' \circ \phi = \Phi \circ s \quad (2.7)$$

$$t' \circ \phi = \Phi \circ t \quad (2.8)$$

$$\gamma' \circ \Phi = p \circ (\Phi \otimes_B \Phi) \circ \gamma \quad (2.9)$$

$$\pi' \circ \Phi = \phi \circ \pi. \quad (2.10)$$

where in (2.9) A' is regarded as a B - B bimodule via ϕ and $p : A' \otimes_B A' \rightarrow A' \otimes_{B'} A'$ is the canonical epimorphism.

The bimodule ${}_B A_B$, appearing in Definition 2.1, is defined in terms of multiplication on the left. Hence – following the terminology of [19] – we use the name *left* bialgebroid for this structure. In terms of right multiplication one defines right bialgebroids analogously. For the details we refer to [19].

Once the map $\gamma : A \rightarrow A \otimes_B A$ is given, we can define $\gamma^{op} : A \rightarrow A \otimes_{B^{op}} A$ via $a \mapsto a_{(2)} \otimes a_{(1)}$. It is straightforward to check that if $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ is a left bialgebroid then $\mathcal{A}_{L^{cop}} = (A, B^{op}, t, s, \gamma^{op}, \pi)$ is also a left bialgebroid and $\mathcal{A}_L^{op} = (A^{op}, B, t, s, \gamma, \pi)$ is a right bialgebroid.

In the case of a left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$, the category ${}_A \mathcal{M}$ of left A -modules is a monoidal category. As a matter of fact, any left A -module is a B - B bimodule via s and t . The monoidal product in ${}_A \mathcal{M}$ is defined as the B -module tensor product with A -module structure

$$a \cdot (m \otimes_B m') := a_{(1)} \cdot m \otimes_B a_{(2)} \cdot m' \quad \text{for } a \in A, m \otimes_B m' \in M \otimes_B M'.$$

Just the same way as axiom (2.3), also this definition makes sense in view of (2.1). The monoidal unit is B with the A -module structure

$$a \cdot b := \pi(as(b)) \quad \text{for } a \in A, b \in B.$$

Analogously, in the case of a right bialgebroid \mathcal{A}_R , the category \mathcal{M}_A of right A -modules is a monoidal category.

The B -coring structure $({}_B A_B, \gamma, \pi)$, underlying a left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$, gives rise to a k -algebra structure on any of the B -duals of ${}_B A_B$ ([11], 17.8). The multiplication on the k -module $*\mathcal{A} := {}_B \text{Hom}(A, B)$, for example, is given by

$$(*\phi * \psi)(a) = * \psi(t \circ * \phi(a_{(2)}) a_{(1)}) \quad \text{for } * \phi, * \psi \in *\mathcal{A}, a \in A \quad (2.11)$$

and the unit is π . $*\mathcal{A}$ is a left A -module and A is a right $*\mathcal{A}$ -module via

$$a \rightarrow * \phi := * \phi(_ a) \quad \text{and} \quad a \leftarrow * \phi := t \circ * \phi(a_{(2)}) a_{(1)} \quad (2.12)$$

for $* \phi \in *\mathcal{A}$, $a \in A$. As it is well known [40, 19], $*\mathcal{A}$ is also a $B \otimes_k B^{op}$ -ring via the inclusions

$$\begin{aligned} *s : B &\rightarrow *\mathcal{A} & b &\mapsto \pi(_)b \\ *t : B^{op} &\rightarrow *\mathcal{A} & b &\mapsto \pi(_ s(b)). \end{aligned}$$

Both maps $*s$ and $*t$ are split injections of B -modules with common left inverse $*\pi : *\mathcal{A} \rightarrow B$, $*\phi \mapsto *\phi(1_A)$. What is more, if A is finitely generated and projective as a left B -module, then $*\mathcal{A}$ has also a right bialgebroid structure (with source and target maps $*s$ and $*t$, respectively, and counit $*\pi$).

Notice that the algebra $*\mathcal{A}$ reduces to the opposite of the usual dual algebra if $({}_B A_B, \gamma, \pi)$ is a coalgebra over a commutative ring B . In the case when A is a finitely generated projective left B -module, also the coproduct specializes to the opposite of the usual one in the case when \mathcal{A} is a bialgebra. This convention is responsible for duality to flip the notions of left- and right bialgebroids.

Applying the above formulae to the left bialgebroid $(\mathcal{A}_L)_{cop}$, we obtain a $B \otimes_k B^{op}$ -ring structure on $\mathcal{A}_* := \text{Hom}_B(A, B)$. The inclusions $B \rightarrow \mathcal{A}_*$ and $B^{op} \rightarrow \mathcal{A}_*$ will be denoted by s_* and t_* , respectively. In particular, \mathcal{A}_* is a left A -module and A is a right \mathcal{A}_* -module via

$$a \rightharpoonup \phi_* := (_ a) \quad \text{and} \quad a \leftharpoonup \phi_* := s \circ \phi_*(a_{(1)}) a_{(2)}. \quad (2.13)$$

If the module A is finitely generated and projective as a right B -module then \mathcal{A}_* is also a right bialgebroid.

In the case of a right bialgebroid $\mathcal{A}_R = (A, B, s, t, \gamma, \pi)$, application of the opposite of the multiplication formula (2.11) to $(\mathcal{A}_R)_{cop}^{op}$ and to $(\mathcal{A}_R)^{op}$ results in $B \otimes_k B^{op}$ -ring structures on $\mathcal{A}^* := \text{Hom}_B(A, B)$ and $*\mathcal{A} := {}_B \text{Hom}(A, B)$, respectively. We obtain inclusions $s^* : B \rightarrow \mathcal{A}^*$, $t^* : B^{op} \rightarrow \mathcal{A}^*$, $*s : B \rightarrow *\mathcal{A}$ and $*t : B^{op} \rightarrow *\mathcal{A}$.

In particular, \mathcal{A}^* and $*\mathcal{A}$ are right A -modules and A is a left \mathcal{A}^* -module and a left $*\mathcal{A}$ -module via the formulae

$$\phi^* \leftharpoonup a := \phi^*(a _) \quad \text{and} \quad \phi^* \rightharpoonup a := a^{(2)} t \circ \phi^*(a^{(1)}) \quad (2.14)$$

$$*\phi \leftharpoonup a := *\phi(a _) \quad \text{and} \quad *\phi \rightharpoonup a := a^{(1)} s \circ *\phi(a^{(2)}) \quad (2.15)$$

for $\phi^* \in \mathcal{A}^*$, $*\phi \in *\mathcal{A}$ and $a \in A$. If A is finitely generated and projective as a right, or as a left B -module then the corresponding dual is also a left bialgebroid.

Before defining the structure that is going to be the subject of the paper, let us stop here and introduce some notations. Analogous notations were already used in [6].

When dealing with a $B \otimes_k B^{op}$ -ring A , we have to face the situation that A carries different module structures over the base algebra B . In this situation the usual notation $A \otimes_B A$ would be ambiguous. Therefore we make the following notational convention. In terms of the maps $s : B \rightarrow A$ and $t : B^{op} \rightarrow A$, we introduce four B -modules

$$\begin{aligned} {}_B A : & \quad b \cdot a := s(b)a \\ A_B : & \quad a \cdot b := t(b)a \\ A^B : & \quad a \cdot b := as(b) \\ {}^B A : & \quad b \cdot a := at(b). \end{aligned} \quad (2.16)$$

(Our notation can be memorized as left indices stand for left modules and right indices stand for right modules. Upper indices are used to label modules defined in terms of right multiplication and lower indices are used for modules defined in terms of left multiplication.)

In writing B -module tensor products, we write out explicitly the module structures of the factors that are taking part in the tensor products, and do not put marks under the symbol \otimes . E.g. we write $A_B \otimes_B A$. Normally we do not denote the module structures that are not taking part in the tensor product, this should be clear from the context. In writing elements of tensor product modules we do not distinguish between the various module tensor products. That is, we write both $a \otimes a' \in A_B \otimes_B A$ and $c \otimes c' \in A^B \otimes_B A$, for example.

A left B -module can be considered as a right B^{op} -module, and sometimes we want to take a module tensor product over B^{op} . In this case we use the name of the corresponding B -module and the fact that the tensor product is taken over B^{op} should be clear from the order of the factors. For example, ${}_B A \otimes_B A$ is the B^{op} -module tensor product of the right B^{op} module defined via

multiplication by $s(b)$ on the left, and the left B^{op} -module defined via multiplication by $t(b)$ on the left.

In writing multiple tensor products, we use different types of letters to denote which module structures take part in the same tensor product. For example, the B -module tensor product $A_B \otimes^B A$ can be given a right B module structure via multiplication by $t(b)$ on the left in the second factor. The tensor product of this right B -module with ${}_B A$ is denoted by $A_B \otimes^B A_B \otimes_{{}_B A}$.

We are ready to introduce the structure that is the main subject of the paper:

Definition 2.2. A *Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ consists of a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$, a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ on the *same* total algebra A , and a k -module map $S : A \rightarrow A$, called the antipode, such that the following axioms hold true:

$$\begin{aligned} i) \quad & s_L \circ \pi_L \circ t_R = t_R, \quad t_L \circ \pi_L \circ s_R = s_R \quad \text{and} \\ & s_R \circ \pi_R \circ t_L = t_L, \quad t_R \circ \pi_R \circ s_L = s_L \end{aligned} \quad (2.17)$$

$$\begin{aligned} ii) \quad & (\gamma_L \otimes^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L \quad \text{as maps } A \rightarrow A_L \otimes_L A^R \otimes^R A \quad \text{and} \\ & (\gamma_R \otimes_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R \quad \text{as maps } A \rightarrow A^R \otimes^R A_L \otimes_L A \end{aligned} \quad (2.18)$$

$$\begin{aligned} iii) \quad & S \text{ is both an } L\text{-}L \text{ bimodule map } {}^L A_L \rightarrow {}_L A^L \text{ and an } R\text{-}R \text{ bimodule map} \\ & {}^R A_R \rightarrow {}_R A^R \end{aligned} \quad (2.19)$$

$$\begin{aligned} iv) \quad & m_A \circ (S \otimes_L A) \circ \gamma_L = s_R \circ \pi_R \quad \text{and} \\ & m_A \circ (A^R \otimes S) \circ \gamma_R = s_L \circ \pi_L. \end{aligned} \quad (2.20)$$

If $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid then so is $\mathcal{A}_{cop}^{op} = ((\mathcal{A}_R)_{cop}^{op}, (\mathcal{A}_L)_{cop}^{op}, S)$ and if S is bijective then also $\mathcal{A}_{cop} = ((\mathcal{A}_L)_{cop}, (\mathcal{A}_R)_{cop}, S^{-1})$ and $\mathcal{A}^{op} = ((\mathcal{A}_R)^{op}, (\mathcal{A}_L)^{op}, S^{-1})$.

The following modification of Sweedler's convention will turn out to be useful. For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we use the notation $\gamma_L(a) = a_{(1)} \otimes a_{(2)}$ with lower indices, and $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$ with upper indices for $a \in A$ in the case of the coproducts of \mathcal{A}_L and of \mathcal{A}_R , respectively. The axioms (2.18) read in this notation as

$$\begin{aligned} a^{(1)}_{(1)} \otimes a^{(1)}_{(2)} \otimes a^{(2)} &= a_{(1)} \otimes a_{(2)}^{(1)} \otimes a_{(2)}^{(2)} \\ a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes a_{(2)} &= a^{(1)} \otimes a^{(2)}_{(1)} \otimes a^{(2)}_{(2)} \end{aligned}$$

for $a \in A$.

Examples of Hopf algebroids (with bijective antipode) are collected in [6].

Proposition 2.3. 1) The base algebras L and R of the left and right bialgebroids in a Hopf algebroid are anti-isomorphic.

2) For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, the pair $(S, \pi_L \circ s_R)$ is a left bialgebroid homomorphism $(\mathcal{A}_R)_{cop}^{op} \rightarrow \mathcal{A}_L$ and $(S, \pi_R \circ s_L)$ is a left bialgebroid homomorphism $\mathcal{A}_L \rightarrow (\mathcal{A}_R)_{cop}^{op}$.

Proof. 1): Both $\pi_R \circ s_L$ and $\pi_R \circ t_L$ are anti-isomorphisms $L \rightarrow R$ with inverses $\pi_L \circ t_R$ and $\pi_L \circ s_R$, respectively.

2): By part 1), the map $\pi_L \circ s_R : R^{op} \rightarrow L$ is an algebra homomorphism. It follows from (2.19), (2.20) and some bialgebroid identities that $S : A^{op} \rightarrow A$ is an algebra homomorphism, as for $a, b \in A$ we have

$$\begin{aligned} S(1_A) &= 1_A \quad S(1_A) = s_L \circ \pi_L(1_A) = 1_A \quad \text{and} \\ S(ab) &= S[t_L \circ \pi_L(a_{(2)}) \, a_{(1)} \, b] \\ &= S[a_{(1)} \, t_L \circ \pi_L(b_{(2)}) \, b_{(1)}] \, a_{(2)}^{(1)} S(a_{(2)}^{(2)}) \\ &= S(a_{(1)} b_{(1)}) a_{(2)}^{(1)} b_{(2)}^{(1)} S(b_{(2)}^{(2)}) S(a_{(2)}^{(2)}) \\ &= S[a^{(1)}_{(1)} b^{(1)}_{(1)}] \, a^{(1)}_{(2)} b^{(1)}_{(2)} S(b^{(2)}) S(a^{(2)}) \\ &= s_R \circ \pi_R(a^{(1)} b^{(1)}) \, S(b^{(2)}) \, S(a^{(2)}) \\ &= S \left[b^{(2)} \, t_R \circ \pi_R \left(t_R \circ \pi_R(a^{(1)}) \, b^{(1)} \right) \right] S(a^{(2)}) \\ &= S(b) \, s_R \circ \pi_R(a^{(1)}) \, S(a^{(2)}) = S(b) S(a). \end{aligned}$$

In the verification of the anti-multiplicativity of S , the third equality follows by axioms (2.1) and (2.20). In the fourth equality we used that, for any $x, y \in A$, there are well defined maps $A_L \otimes_L A^R \otimes^R A \rightarrow A$, $a \otimes b \otimes c \mapsto S(ax)by^{(1)}S(y^{(2)})S(c)$ and $a \otimes b \otimes c \mapsto S(x_{(1)}a)x_{(2)}bS(c)S(y)$, that can be composed with the equal maps $(\gamma_L \otimes^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L : A \rightarrow A_L \otimes_L A^R \otimes^R A$, cf. (2.18). The fifth equality follows by (2.3) and (2.20), the sixth equality is a consequence of (2.19) and the right bialgebroid analogue of (2.6). The last two equalities both follow by the counitality of γ_R and (2.19), taking into account that $\gamma_R(t_R(r)b) = t_R(r)^{(1)}b^{(1)} \otimes t_R(r)^{(2)}b^{(2)} = t_R(r)b^{(1)} \otimes b^{(2)}$, for all $r \in R$ and $b \in A$.

Properties (2.7-2.8) follow from (2.19) and (2.17) as

$$s_L \circ \pi_L \circ s_R = S \circ t_L \circ \pi_L \circ s_R = S \circ s_R \quad (2.21)$$

$$t_L \circ \pi_L \circ s_R = s_R = S \circ t_R. \quad (2.22)$$

Properties (2.9-2.10) are checked on an element $a \in A$ as

$$\begin{aligned} \gamma_L \circ S(a) &= S(a^{(1)}_{(1)}a^{(1)}_{(2)}S(a^{(2)}) \otimes S(a^{(1)}_{(1)}a^{(1)}_{(2)}) \\ &= S(a^{(1)}_{(1)}a^{(1)}_{(2)})^{(1)}S(a^{(2)}) \otimes S(a^{(1)}_{(1)}a^{(1)}_{(2)})^{(1)}S(a^{(1)}_{(2)}a^{(2)}) \\ &= \left(S(a^{(1)}_{(1)}a^{(1)}_{(2)})^{(1)} \right)_{(1)} S(a^{(2)}) \otimes \left(S(a^{(1)}_{(1)}a^{(1)}_{(2)})^{(1)} \right)_{(2)} S(a^{(1)}_{(2)}a^{(2)}) \\ &= (S \otimes S) \circ \gamma_R^{op}(a) \end{aligned} \quad (2.23)$$

$$\pi_L \circ S(a) = \pi_L[S(a_{(1)}) s_L \circ \pi_L(a_{(2)})] = \pi_L[S(a_{(1)})a_{(2)}] = \pi_L \circ s_R \circ \pi_R(a). \quad (2.24)$$

In order to check the compatibility condition between S and the coproducts, note that there is a well defined map $A_L \otimes_L A^R \otimes^R A \rightarrow A_L \otimes_L A$, $a \otimes b \otimes c \mapsto a \otimes bS(c)$. Composing it with the equal maps $(\gamma_L \otimes^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L : A \rightarrow A_L \otimes_L A^R \otimes^R A$ (cf. (2.18)), and using (2.20), we conclude that

$$a^{(1)}_{(1)} \otimes a^{(1)}_{(2)} S(a^{(2)}) = a \otimes 1_A, \quad \text{for all } a \in A. \quad (2.25)$$

Applying to both sides of (2.25) the well defined map $A_L \otimes_L A \rightarrow A_L \otimes_L A$, $a \otimes b \mapsto S(a)_{(1)}b \otimes S(a)_{(2)}$, we conclude on the first equality of the computation in (2.23). In the second equality we used (2.25) again. The third equality follows by multiplicativity of γ_L , cf. (2.3). The last equality is derived similarly to the first one: There is a well defined map $A_L \otimes_L A^R \otimes^R A \rightarrow A^R \otimes^R A$, $a \otimes b \otimes c \mapsto S(a)b \otimes S(c)$. Composing it with the equal maps $(\gamma_L \otimes^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L : A \rightarrow A_L \otimes_L A^R \otimes^R A$, using (2.20) and the identity $S \circ t_R = s_R$, we conclude that $S(a_{(1)})a_{(2)}^{(1)} \otimes S(a_{(2)})^{(2)} = 1 \otimes S(a)$, for all $a \in A$. Applying to both sides of this identity the well defined map $A^R \otimes^R A \rightarrow A_L \otimes_L A$, $a \otimes b \mapsto a_{(1)} \otimes a_{(2)}b$, we obtain $1_A \otimes S(a) = (S(a_{(1)})a_{(2)}^{(1)})_{(1)} \otimes (S(a_{(1)})a_{(2)}^{(1)})_{(2)} S(a_{(2)})^{(2)}$, that explains the last equality in (2.23).

In the first equality of (2.24), we used the counitality of γ_L and (2.19). The second equality follows by (2.5). To derive the last equality, we made use of (2.20).

The proof is completed by the observation that in passing from the Hopf algebroid \mathcal{A} to \mathcal{A}_{cop}^{op} the roles of $(S, \pi_L \circ s_R)$ and $(S, \pi_R \circ s_L)$ become interchanged. \blacksquare

Proposition 2.4. *The left bialgebroid \mathcal{A}_L in a Hopf algebroid \mathcal{A} is a \times_L -Hopf algebra in the sense of [31]. That is, the map*

$$\alpha : {}^L A \otimes A_L \rightarrow A_L \otimes {}^L A \quad a \otimes b \mapsto a_{(1)} \otimes a_{(2)}b$$

is bijective.

Proof. The inverse of α is given by

$$\alpha^{-1} : A_L \otimes {}^L A \rightarrow {}^L A \otimes A_L \quad a \otimes b \mapsto a^{(1)} \otimes S(a^{(2)})b. \quad \blacksquare$$

The relation between the left and the right bialgebroids in a Hopf algebroid \mathcal{A} implies relations between the dual algebras $\mathcal{A}^* \equiv \text{Hom}_R(A^R, R)$ and $\mathcal{A}_* \equiv \text{Hom}_L(A_L, L)$ and also between ${}^* \mathcal{A} \equiv {}_R \text{Hom}({}^R A, R)$ and ${}_* \mathcal{A} \equiv {}_L \text{Hom}({}_L A, L)$:

Lemma 2.5. For a Hopf algebroid \mathcal{A} , there exist algebra anti-isomorphisms $\sigma : {}^*\mathcal{A} \rightarrow {}^*\mathcal{A}$ and $\chi : \mathcal{A}^* \rightarrow \mathcal{A}_*$ satisfying

$$a \leftharpoonup {}^*\phi = \sigma({}^*\phi) \rightharpoonup a \quad \text{and} \quad (2.26)$$

$$\phi^* \rightharpoonup a = a \leftharpoonup \chi(\phi^*) \quad (2.27)$$

for all ${}^*\phi \in {}^*\mathcal{A}$, $\phi^* \in \mathcal{A}^*$ and $a \in A$.

Proof. We leave it to the reader to check that the maps

$$\begin{aligned} \sigma : {}^*\mathcal{A} &\rightarrow {}^*\mathcal{A} & {}^*\phi &\mapsto \pi_R(_ \leftharpoonup {}^*\phi) \quad \text{and} \\ \chi : \mathcal{A}^* &\rightarrow \mathcal{A}_* & \phi^* &\mapsto \pi_L(\phi^* \rightharpoonup _) \end{aligned}$$

are algebra anti-homomorphisms satisfying (2.26-2.27). The inverses are given by

$$\begin{aligned} \sigma^{-1} : {}^*\mathcal{A} &\rightarrow {}^*\mathcal{A} & {}^*\phi &\mapsto \pi_L({}^*\phi \rightharpoonup _) \quad \text{and} \\ \chi^{-1} : \mathcal{A}_* &\rightarrow \mathcal{A}^* & \phi_* &\mapsto \pi_R(_ \leftharpoonup \phi_*). \end{aligned} \quad \blacksquare$$

Lemma 2.6. For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ with a bijective antipode S , the following assertions hold:

- 1) The module A_L is finitely generated and projective if and only if the module ${}^R A$ is finitely generated and projective.
- 2) The module ${}_L A$ is finitely generated and projective if and only if the module A^R is finitely generated and projective.

Proof. 1): In terms of the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module A_L , the dual bases, $\{k_j\} \subset A$ and $\{\kappa_j^*\} \subset {}^*\mathcal{A}$ for the module ${}^R A$, can be constructed by the requirement that

$$\sum_j {}^*\kappa_j \otimes k_j = \sum_i \pi_R \circ t_L \circ \beta_*^i \circ S \otimes S^{-1}(b_i) \quad \text{as elements of } {}^*\mathcal{A}_R \otimes {}^R A.$$

The converse implication follows by applying the same reasoning to \mathcal{A}_{cop}^{op} .

2) follows by applying part 1) to the Hopf algebroid \mathcal{A}^{op} . ■

Now we turn to the study of the notion of integrals in Hopf algebroids. For a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ and a left A -module M , the *invariants* of M with respect to \mathcal{A}_L are the elements of

$$\text{Inv}(M) := \{ n \in M \mid a \cdot n = s_L \circ \pi_L(a) \cdot n \quad \forall a \in A \}.$$

Clearly, the invariants of M with respect to $(\mathcal{A}_L)_{cop}$ coincide with its invariants with respect to \mathcal{A}_L . The invariants of a right A -module M with respect to a right bialgebroid \mathcal{A}_R are defined as the invariants of M (viewed as a left A^{op} -module) with respect to $(\mathcal{A}_R)^{op}$.

Definition 2.7. The *left integrals* in a left bialgebroid \mathcal{A}_L are the invariants of the left regular A -module with respect to \mathcal{A}_L .

The *right integrals* in a right bialgebroid \mathcal{A}_R are the invariants of the right regular A -module with respect to \mathcal{A}_R .

The *left/right integrals* in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are the left/right integrals in $\mathcal{A}_L/\mathcal{A}_R$, that is, the elements of

$$\begin{aligned} \mathcal{L}(\mathcal{A}) &= \{ \ell \in A \mid a\ell = s_L \circ \pi_L(a) \ell \quad \forall a \in A \} \quad \text{and} \\ \mathcal{R}(\mathcal{A}) &= \{ \wp \in A \mid \wp a = \wp s_R \circ \pi_R(a) \quad \forall a \in A \}. \end{aligned}$$

For any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, we have $\mathcal{L}(\mathcal{A}) = \mathcal{R}(\mathcal{A}_{cop}^{op})$ and if S is bijective then also $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{cop}) = \mathcal{R}(\mathcal{A}^{op})$. Since for $\ell \in \mathcal{L}(\mathcal{A})$ and $a \in A$,

$$S(\ell)a = S[t_L \circ \pi_L(a_{(1)}) \ell]a_{(2)} = S(a_{(1)}\ell)a_{(2)} = S(\ell) s_R \circ \pi_R(a),$$

we have $S(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{R}(\mathcal{A})$ and, similarly, $S(\mathcal{R}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A})$.

Scholium 2.8. *The following properties of an element $\ell \in A$ are equivalent:*

- 1.a) $\ell \in \mathcal{L}(\mathcal{A})$
- 1.b) $S(a)\ell^{(1)} \otimes \ell^{(2)} = \ell^{(1)} \otimes a\ell^{(2)} \quad \forall a \in A$
- 1.c) $a\ell^{(1)} \otimes S(\ell^{(2)}) = \ell^{(1)} \otimes S(\ell^{(2)})a \quad \forall a \in A.$

The following properties of the element $\wp \in A$ are also equivalent:

- 2.a) $\wp \in \mathcal{R}(\mathcal{A})$
- 2.b) $\wp_{(1)} \otimes \wp_{(2)}S(a) = \wp_{(1)}a \otimes \wp_{(2)} \quad \forall a \in A$
- 2.c) $S(\wp_{(1)}) \otimes \wp_{(2)}a = aS(\wp_{(1)}) \otimes \wp_{(2)} \quad \forall a \in A.$

By comodules over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ we mean comodules over the L -coring $({}_L A_L, \gamma_L, \pi_L)$, and by comodules over a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ we mean comodules over the R -coring $({}^R A^R, \gamma_R, \pi_R)$. The pair $({}_L A, \gamma_L)$ is a left comodule, and (A_L, γ_L) is a right comodule over the left bialgebroid \mathcal{A}_L . Since the L -coring $({}_L A_L, \gamma_L, \pi_L)$ possesses a grouplike element 1_A , also (L, s_L) is a left comodule and (L, t_L) is a right comodule over \mathcal{A}_L (see [11], 28.2). Similarly, (A^R, γ_R) and (R, s_R) are right comodules, and $({}^R A, \gamma_R)$ and (R, t_R) are left comodules over \mathcal{A}_R .

Definition 2.9. An s -integral on a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left \mathcal{A}_L -comodule map $*\rho : ({}_L A, \gamma_L) \rightarrow (L, s_L)$. That is, an element of

$$\mathcal{R}(*\mathcal{A}) := \{ *\rho \in *\mathcal{A} \mid (A_L \otimes *\rho) \circ \gamma_L = s_L \circ *\rho \}.$$

A t -integral on \mathcal{A}_L is a right \mathcal{A}_L -comodule map $(A_L, \gamma_L) \rightarrow (L, t_L)$. That is, an element of

$$\mathcal{R}(\mathcal{A}_*) := \{ \rho_* \in \mathcal{A}_* \mid (\rho_* \otimes {}_L A) \circ \gamma_L = t_L \circ \rho_* \}.$$

An s -integral on a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ is a right \mathcal{A}_R -comodule map $(A^R, \gamma_R) \rightarrow (R, s_R)$. That is, an element of

$$\mathcal{L}(\mathcal{A}^*) := \{ \lambda^* \in \mathcal{A}^* \mid (\lambda^* \otimes {}^R A) \circ \gamma_R = s_R \circ \lambda^* \}.$$

A t -integral on \mathcal{A}_R is a left \mathcal{A}_R -comodule map $({}^R A, \gamma_R) \rightarrow (R, t_R)$. That is, an element of

$$\mathcal{L}(*\mathcal{A}) := \{ *\lambda \in *\mathcal{A} \mid (A^R \otimes *\lambda) \circ \gamma_R = t_R \circ *\lambda \}.$$

The *right/left s - and t -integrals on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$* are the s - and t -integrals on $\mathcal{A}_L/\mathcal{A}_R$.

The integrals on a *left/right* bialgebroid are checked to be invariants of the appropriate *right/left* regular module – justifying our usage of the terms ‘*right*’ and ‘*left*’ integrals for them (cf. the remark in Section 2 about our using the opposite - co-opposite of the convention usual in the case of bialgebras, when defining the dual bialgebroids $*\mathcal{A}$ and \mathcal{A}^*). As a matter of fact, for example, if $*\rho \in \mathcal{R}(*\mathcal{A})$ then

$$[*\rho * \phi](a) = *\phi(a \leftarrow *\rho) = *\phi(s_L \circ *\rho(a)) = *\rho(a) *\phi(1_A) = [*\rho * s \circ \pi(*\phi)](a) \quad (2.28)$$

for all $*\phi \in *\mathcal{A}$ and $a \in A$. If the module ${}_L A$ is finitely generated and projective (hence $*\mathcal{A}$ is a right bialgebroid) then also the converse is true, so in this case the s -integrals on \mathcal{A}_L are the same as the right integrals in $*\mathcal{A}$. Similar statements hold true on the elements of $\mathcal{R}(\mathcal{A}_*)$, $\mathcal{L}(\mathcal{A}^*)$ and $\mathcal{L}(*\mathcal{A})$.

The reader should be warned that integrals on Hopf algebras H over commutative rings k are defined in the literature sometimes as comodule maps $H \rightarrow k$ – similarly to our Definition 2.9 –, sometimes by the analogue of the weaker invariant condition (2.28).

For any Hopf algebroid \mathcal{A} , we have $\mathcal{R}(*\mathcal{A}) = \mathcal{L}((\mathcal{A}_{cop}^{op})^*)$ and $\mathcal{R}(\mathcal{A}_*) = \mathcal{L}^*(\mathcal{A}_{cop}^{op})$. If the antipode is bijective then also $\mathcal{R}(*\mathcal{A}) = \mathcal{R}((\mathcal{A}_{cop})_*) = \mathcal{L}^*(\mathcal{A}^{op})$.

Scholium 2.10. Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid. The following properties of an element ${}_*\rho \in {}_*\mathcal{A}$ are equivalent:

- 1.a) ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$
- 1.b) $\pi_R \circ s_L \circ {}_*\rho \in \mathcal{L}({}_*\mathcal{A})$
- 1.c) $s_L \circ {}_*\rho(aS(b_{(1)})) \ b_{(2)} = t_L \circ {}_*\rho(a_{(2)}S(b)) \ a_{(1)} \quad \forall a, b \in A.$

The following properties of an element $\rho_* \in \mathcal{A}_*$ are equivalent:

- 2.a) $\rho_* \in \mathcal{R}(\mathcal{A}_*)$
- 2.b) $\pi_R \circ t_L \circ \rho_* \in \mathcal{L}(\mathcal{A}^*)$
- 2.c) $t_L \circ \rho_*(ab^{(1)}) \ S(b^{(2)}) = s_L \circ \rho_*(a_{(1)}b) \ a_{(2)} \quad \forall a, b \in A.$

The following properties of an element $\lambda^* \in \mathcal{A}^*$ are equivalent:

- 3.a) $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$
- 3.b) $\pi_L \circ s_R \circ \lambda^* \in \mathcal{R}(\mathcal{A}_*)$
- 3.c) $a^{(1)} \ s_R \circ \lambda^*(S(a^{(2)})b) = b^{(2)} \ t_R \circ \lambda^*(S(a)b^{(1)}) \quad \forall a, b \in A.$

The following properties of an element ${}^*\lambda \in {}^*\mathcal{A}$ are equivalent:

- 4.a) ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$
- 4.b) $\pi_L \circ t_R \circ {}^*\lambda \in \mathcal{R}({}_*\mathcal{A})$
- 4.c) $S(a_{(1)}) \ t_R \circ {}^*\lambda(a_{(2)}b) = b^{(1)} \ s_R \circ {}^*\lambda(ab^{(2)}) \quad \forall a, b \in A.$

In particular, for ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ the element ${}_*\rho \circ S$ belongs to $\mathcal{R}(\mathcal{A}_*)$ and for $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ the element $\lambda^* \circ S$ belongs to $\mathcal{L}({}_*\mathcal{A})$.

3 Maschke type theorems

The most classical version of Maschke's theorem [23] considers group algebras over fields. It states that the group algebra of a finite group G over a field F is semi-simple if and only if the characteristic of F does not divide the order of G . This result has been generalized to finite dimensional Hopf algebras H over fields F by Sweedler [33] proving that H is a separable F -algebra if and only if it is semi-simple and if and only if there exists a normalized left integral in H . The proof goes as follows. It is a classical result that a separable algebra over a field is semi-simple. If H is semi-simple then, in particular, the H -module on F , defined in terms of the counit, is projective. This means that the counit, as an H -module map $H \rightarrow F$, splits. Its right inverse maps the unit of F into a normalized integral. Finally, in terms of a normalized integral one can construct an H -bilinear right inverse for the multiplication map $H \otimes_F H \rightarrow H$.

The only difficulty in the generalization of Maschke's theorem to Hopf algebras over commutative rings comes from the fact that in the case of an algebra A over a commutative base ring k , separability does not imply the semi-simplicity of A in the sense [29] that every (left or right) A -module was projective. It implies [16, 17], however, that every A -module is (A, k) -projective, i.e. that every epimorphism of A -modules which is k -split, is also A -split. In order to avoid confusion, we will say that the k -algebra A is *semi-simple* [29] if it is an Artinian semi-simple ring i.e. if any A -module is projective. By the terminology of [16] we call A a (left or right) *semi-simple extension* of k if any (left or right) A -module is (A, k) -projective.

Since the counit of a Hopf algebra H over a commutative ring k is a split epimorphism of k -modules, Maschke's theorem generalizes to this case in the following form [13, 21]. The extension $k \rightarrow H$ is separable if and only if it is (left and right) semi-simple and if and only if there exist normalized (left and right) integrals in H .

In this section we investigate the properties of the total algebra of a Hopf algebroid as an extension of the base algebra, that are equivalent to the existence of normalized integrals *in* the Hopf algebroid. Dually, we investigate also the properties of the coring over the base algebra underlying a Hopf algebroid, that are equivalent to the existence of normalized integrals *on* the Hopf algebroid (in any of the four possible senses).

A Maschke type theorem on certain Hopf algebroids can be obtained also by application of ([38], Theorem 4.2). Notice, however, that the Hopf algebroids occurring this way are only the Frobenius Hopf algebroids (discussed in Section 4 below), that is the Hopf algebroids possessing non-degenerate integrals (which are called Frobenius integrals in [38]).

Following Theorem 3.1 generalizes results from ([13], Proposition 4.7) and ([21], Theorem 3.3).

Theorem 3.1. (*Maschke Theorem for Hopf algebroids.*) *The following assertions on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent:*

1.a) *The extension $s_R : R \rightarrow A$ is separable. That is, the multiplication map $A^R \otimes_R A \rightarrow A$ splits as an A - A bimodule epimorphism.*

1.b) *The extension $t_R : R^{op} \rightarrow A$ is separable. That is, the multiplication map ${}^R A \otimes A_R \rightarrow A$ splits as an A - A bimodule epimorphism.*

1.c) *The extension $s_L : L \rightarrow A$ is separable. That is, the multiplication map $A^L \otimes_L A \rightarrow A$ splits as an A - A bimodule epimorphism.*

1.d) *The extension $t_L : L^{op} \rightarrow A$ is separable. That is, the multiplication map ${}^L A \otimes A_L \rightarrow A$ splits as an A - A bimodule epimorphism.*

2.a) *The extension $s_R : R \rightarrow A$ is right semi-simple. That is, any right A -module is (A, R) -projective.*

2.b) *The extension $t_R : R^{op} \rightarrow A$ is right semi-simple. That is, any right A -module is (A, R^{op}) -projective.*

2.c) *The extension $s_L : L \rightarrow A$ is left semi-simple. That is, any left A -module is (A, L) -projective.*

2.d) *The extension $t_L : L^{op} \rightarrow A$ is left semi-simple. That is, any left A -module is (A, L^{op}) -projective.*

3.a) *There exists a normalized left integral in \mathcal{A} . That is, an element $\ell \in \mathcal{L}(\mathcal{A})$ such that $\pi_L(\ell) = 1_L$.*

3.b) *There exists a normalized right integral in \mathcal{A} . That is, an element $\wp \in \mathcal{R}(\mathcal{A})$ such that $\pi_R(\wp) = 1_R$.*

4.a) *The epimorphism $\pi_R : A \rightarrow R$ splits as a right A -module map.*

4.b) *The epimorphism $\pi_L : A \rightarrow L$ splits as a left A -module map.*

Proof. 1.a) \Rightarrow 2.a), 1.b) \Rightarrow 2.b), 1.c) \Rightarrow 2.c) and 1.d) \Rightarrow 2.d): It is proven in ([17], Proposition 2.6) that a separable extension is both left- and right semi-simple.

2.a) \Rightarrow 4.a) (2.b) \Rightarrow 4.a): The epimorphism π_R is split as a right (left) R -module map by s_R (by t_R), hence it is split as a right A -module map.

4.a) \Rightarrow 3.b): Let $\nu : R \rightarrow A$ be the right inverse of π_R in \mathcal{M}_A . Then $\wp := \nu(1_R)$ is a normalized right integral in \mathcal{A} .

3.a) \Leftrightarrow 3.b): By part 2) of Proposition 2.3 the antipode takes a normalized left/right integral to a normalized right/left integral.

3.a) \Rightarrow 1.a) and 3.b) \Rightarrow 1.b): If ℓ is a normalized left integral in \mathcal{A} then, by Scholium 2.8, the required right inverse of the multiplication map $A^R \otimes_R A \rightarrow A$ is given by the A - A bimodule map $a \mapsto a\ell^{(1)} \otimes S(\ell^{(2)}) \equiv \ell^{(1)} \otimes S(\ell^{(2)})a$. Similarly, if \wp is a normalized right integral in \mathcal{A} then the right inverse of the multiplication map ${}^R A \otimes A_R \rightarrow A$ is given by $a \mapsto aS(\wp_{(1)}) \otimes \wp_{(2)} \equiv S(\wp_{(1)}) \otimes \wp_{(2)}a$.

The proof is completed by applying the above arguments to the Hopf algebroid \mathcal{A}_{cop}^{op} . \blacksquare

Let us make a comment on the semi-simplicity of the algebra A (cf. [17], Proposition 1.3). If R is a semi-simple algebra and the equivalent conditions of Theorem 3.1 hold true, then A – being a semi-simple extension of a semi-simple algebra – is a semi-simple algebra. On the other

hand, notice that condition 4.a) in Theorem 3.1 is equivalent to the projectivity of the right A -module R . Hence if A is a semi-simple k -algebra then the equivalent conditions of the theorem hold true. It is not true, however, that the semi-simplicity of the total algebra implied the semi-simplicity of the base algebra (which was shown by Lomp to be the case in Hopf algebras [21]). A counterexample can be constructed as follows: If B is a Frobenius algebra over a commutative ring k then $A := \text{End}_k(B)$ has a Hopf algebroid structure over the base B [7]. If B is a Frobenius algebra over a field – which can be non-semi-simple! – then A is a Hopf algebroid with semi-simple total algebra.

Following Theorem 3.2 can be considered as a dual of Theorem 3.1 in the sense that it speaks about corings over the base algebras instead of algebra extensions. It is important to emphasize, however, that the two theorems are independent results. Even in the case of Hopf algebroids such that all module structures (2.16) are finitely generated and projective, the duals are not known to be Hopf algebroids.

Recall that the dual notion of that of a relative projective module is the relative injective comodule. Namely, a comodule M for an R -coring A is called (A, R) -*injective* ([11], 18.18) if any monomorphism of A -comodules from M , which splits as an R -module map, splits also as an A -comodule map.

Theorem 3.2. (*Dual Maschke Theorem for Hopf algebroids.*) *The following assertions on a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent:*

- 1.a) *The R -coring $({}^R A^R, \gamma_R, \pi_R)$ is coseparable. That is, the comultiplication $\gamma_R : A \rightarrow A^R \otimes {}^R A$ splits as an \mathcal{A}_R - \mathcal{A}_R bicomodule monomorphism.*
- 1.b) *The L -coring $({}_L A_L, \gamma_L, \pi_L)$ is coseparable. That is, the comultiplication $\gamma_L : A \rightarrow A_L \otimes {}_L A$ splits as an \mathcal{A}_L - \mathcal{A}_L bicomodule monomorphism.*
- 2.a) *Any right \mathcal{A}_R -comodule is (\mathcal{A}_R, R) -injective.*
- 2.b) *Any left \mathcal{A}_R -comodule is (\mathcal{A}_R, R) -injective.*
- 2.c) *Any left \mathcal{A}_L -comodule is (\mathcal{A}_L, L) -injective.*
- 2.d) *Any right \mathcal{A}_L -comodule is (\mathcal{A}_L, L) -injective.*
- 3.a) *There exists a normalized left s -integral on \mathcal{A} . That is, an element $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ such that $\lambda^*(1_A) = 1_R$.*
- 3.b) *There exists a normalized left t -integral on \mathcal{A} . That is, an element ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$ such that ${}^*\lambda(1_A) = 1_R$.*
- 3.c) *There exists a normalized right s -integral on \mathcal{A} . That is, an element ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ such that ${}_*\rho(1_A) = 1_L$.*
- 3.d) *There exists a normalized right t -integral on \mathcal{A} . That is, an element $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ such that $\rho_*(1_A) = 1_L$.*
- 4.a) *The monomorphism $s_R : R \rightarrow A$ splits as a right \mathcal{A}_R -comodule map.*
- 4.b) *The monomorphism $t_R : R \rightarrow A$ splits as a left \mathcal{A}_R -comodule map.*
- 4.c) *The monomorphism $s_L : L \rightarrow A$ splits as a left \mathcal{A}_L -comodule map.*
- 4.d) *The monomorphism $t_L : L \rightarrow A$ splits as a right \mathcal{A}_L -comodule map.*

Proof. 1.a) \Rightarrow 2.a), 2.b) is proven in ([11], 26.1).

2.a) \Rightarrow 4.a) (2.b) \Rightarrow 4.b)): The monomorphism s_R (t_R) is split as a right (left) R -module map by π_R hence it is split as a right (left) \mathcal{A}_R -comodule map.

4.a) \Rightarrow 3.a) and 4.b) \Rightarrow 3.b): The left inverse λ^* of s_R in the category of right \mathcal{A}_R -comodules is a normalized s -integral on \mathcal{A}_R by very definition. Similarly, the left inverse ${}^*\lambda$ of t_R in the category of left \mathcal{A}_R -comodules is a normalized t -integral on \mathcal{A}_R .

3.a) \Rightarrow 3.b): If λ^* is a normalized s -integral on \mathcal{A}_R then $\lambda^* \circ S$ is a normalized t -integral on \mathcal{A}_R by Scholium 2.10.

3.b) \Rightarrow 1.a): In terms of the normalized t -integral ${}^*\lambda$ on \mathcal{A}_R the required right inverse of the coproduct γ_R is constructed as the map

$$A^R \otimes {}^R A \rightarrow A, \quad a \otimes b \mapsto t_R \circ {}^*\lambda(aS(b_{(1)})) \ b_{(2)}.$$

It is checked to be an \mathcal{A}_R - \mathcal{A}_R bicomodule map using that by Scholium 2.10, 4.b) and 1.c) we have $t_R \circ^* \lambda(aS(b_{(1)})) \ b_{(2)} = a^{(1)} s_R \circ \pi_R[t_R \circ^* \lambda(a^{(2)}S(b_{(1)})) \ b_{(2)}]$ for all a, b in A .

3.a) \Leftrightarrow 3.d) follows from Scholium 2.10, 2.b).

The remaining equivalences are proven by applying the above arguments to the Hopf algebroid \mathcal{A}_{cop}^{op} . \blacksquare

The proofs of Theorem 3.1 and 3.2 can be unified if one formulates them as equivalent statements on the forgetful functors from the category of A -modules, and from the category of \mathcal{A}_L or \mathcal{A}_R -comodules, respectively, to the category of L - or R -modules – as it is done in the case of Hopf algebras over commutative rings in [13]. We believe (together with the referee), however, that the above formulation in terms of algebra extensions and corings, respectively, is more appealing.

4 Frobenius Hopf algebroids and non-degenerate integrals

A left or right integral ℓ in a Hopf algebra (H, Δ, ϵ, S) over a commutative ring k is called non-degenerate [20] if the maps

$$\begin{aligned} \text{Hom}_k(H, k) &\rightarrow H & \phi &\mapsto (\phi \otimes H) \circ \Delta(\ell) \quad \text{and} \\ \text{Hom}_k(H, k) &\rightarrow H & \phi &\mapsto (H \otimes \phi) \circ \Delta(\ell) \end{aligned}$$

are bijective.

The notion of non-degenerate integrals is made relevant by the Larson-Sweedler Theorem [20] stating that a free and finite dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if there exists a non-degenerate left integral in H .

The Larson-Sweedler Theorem has been extended by Pareigis [28] to Hopf algebras over commutative rings with trivial Picard group. He proved also that a bialgebra over an arbitrary commutative ring k , which is a Frobenius k -algebra, is a Hopf algebra if and only if there exists a Frobenius functional $\psi : H \rightarrow k$ satisfying in addition

$$(H \otimes \psi) \circ \Delta = 1_H \psi(_).$$

As a matter of fact, based on the results of [28], the following variant of ([14], 3.2 Theorem 31) can be proven:

Theorem 4.1. *The following properties of a Hopf algebra (H, Δ, ϵ, S) over a commutative ring k are equivalent:*

- 1) H is a Frobenius k -algebra.
- 2) There exists a non-degenerate left integral in H .
- 3) There exists a non-degenerate right integral in H .
- 4) There exists a non-degenerate left integral on H . That is, a Frobenius functional $\psi : H \rightarrow k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(_)$.
- 5) There exists a non-degenerate right integral on H . That is, a Frobenius functional $\psi : H \rightarrow k$ satisfying $(\psi \otimes H) \circ \Delta = 1_H \psi(_)$.

The main subject of the present section is a generalization of Theorem 4.1 to Hopf algebroids.

The most important tool in the proof of Theorem 4.1 is the Fundamental Theorem for Hopf modules [20]. A very general form of it has been proven by Brzeziński ([8], Theorem 5.6, see also [11], 28.19) in the framework of corings. It can be applied in our setting as follows.

Hopf modules over bialgebroids are examples of Doi-Koppinen modules over algebras, studied in [9]. A left-left Hopf module over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left comodule for the comonoid (A, γ_L, π_L) in the category of left A -modules. That is, a pair (M, τ) where M is a left A -module, hence a left L -module ${}_L M$ via s_L . The pair $({}_L M, \tau)$ is a left \mathcal{A}_L -comodule such that $\tau : M \rightarrow A_L \otimes {}_L M$ is a left A -module map to the module

$$a \cdot (b \otimes m) := a_{(1)} b \otimes a_{(2)} \cdot m \quad \text{for } a \in A, \ b \otimes m \in A_L \otimes {}_L M.$$

The right-right Hopf modules over a right bialgebroid \mathcal{A}_R are the left-left Hopf modules over $(\mathcal{A}_R)_{\text{cop}}^{\text{op}}$.

It follows from ([9], Proposition 4.1) that the left-left Hopf modules over \mathcal{A}_L are the left comodules over the A -coring

$$\mathcal{W} := (A_L \otimes_L A, \gamma_L \otimes_L A, \pi_L \otimes_L A), \quad (4.1)$$

where the A - A bimodule structure is given by

$$a \cdot (b \otimes c) \cdot d := a_{(1)} b \otimes a_{(2)} c d \quad \text{for } a, d \in A, \ b \otimes c \in A_L \otimes_L A.$$

The coring (4.1) was studied in [2]. It was shown to possess a group-like element $1_A \otimes 1_A \in A_L \otimes_L A$ and corresponding coinvariant subalgebra $t_L(L)$ in A . The coring (4.1) is Galois (w.r.t. the group-like element $1_A \otimes 1_A$) if and only if \mathcal{A}_L is a \times_L -Hopf algebra in the sense of [31]. Since in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the left bialgebroid \mathcal{A}_L is a \times_L -Hopf algebra by Proposition 2.4, the A -coring (4.1) is Galois in this case. Denote the category of left-left Hopf modules over \mathcal{A}_L (i.e. of left comodules over the coring (4.1)) by ${}^{\mathcal{W}}\mathcal{M}$. The application of ([8], Theorem 5.6) results that if $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid, such that the module ${}^L A$ is faithfully flat, then the functor

$$G : {}^{\mathcal{W}}\mathcal{M} \rightarrow \mathcal{M}_L \quad (M, \tau) \mapsto \text{Coinv}(M)_L := \{ m \in M \mid \tau(m) = 1_A \otimes m \in A_L \otimes_L M \} \quad (4.2)$$

(where the right L -module structure on $\text{Coinv}(M)$ is given via t_L) and the induction functor

$$F : \mathcal{M}_L \rightarrow {}^{\mathcal{W}}\mathcal{M} \quad N_L \mapsto ({}^L A \otimes N_L, \gamma_L \otimes N_L) \quad (4.3)$$

(where the left A -module structure on ${}^L A \otimes N_L$ is given by left multiplication in the first factor) are inverse equivalences.

In the case of Hopf algebras H over commutative rings k , these arguments lead to the Fundamental Theorem only for faithfully flat Hopf algebras. The proof of the Fundamental Theorem in [20], however, does not rely on any assumption on the k -module structure of H . Since the Hopf algebroid structure is more restrictive than the \times_L -Hopf algebra structure, one hopes to prove the Fundamental Theorem for Hopf algebroids also under milder assumptions – using the whole strength of the Hopf algebroid structure.

In the following theorem, Sweedler's index notation $\tau(m) = m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}$ (with implicit summation) is used, for the left coaction $\tau : M \rightarrow \mathcal{W} \otimes_A M \cong A_L \otimes_L M$ of the constituent left L -bialgebroid \mathcal{A}_L in a Hopf algebroid \mathcal{A} , on a left \mathcal{A}_L -comodule M and $m \in M$.

Theorem 4.2. (*Fundamental Theorem for Hopf algebroids.*) *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and \mathcal{W} be the A -coring (4.1). Assume that the kernel of the maps*

$$M \rightarrow A_L \otimes_L M, \quad m \mapsto (m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}) - (1_A \otimes m) \quad (4.4)$$

is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \rightarrow \mathcal{M}_L$, for any $M \in {}^{\mathcal{W}}\mathcal{M}$ (e.g. ${}^L A$ is a flat module). Then the functors $G : {}^{\mathcal{W}}\mathcal{M} \rightarrow \mathcal{M}_L$ in (4.2) and $F : \mathcal{M}_L \rightarrow {}^{\mathcal{W}}\mathcal{M}$ in (4.3) are inverse equivalences.²

Proof. We construct natural isomorphisms $\alpha : F \circ G \rightarrow {}^{\mathcal{W}}\mathcal{M}$ and $\beta : G \circ F \rightarrow \mathcal{M}_L$. The map

$$\alpha_M : {}^L A \otimes \text{Coinv}(M)_L \rightarrow M \quad a \otimes m \mapsto a \cdot m$$

is a left \mathcal{W} -comodule map and natural in M . The isomorphism property is proven by constructing the inverse

$$\alpha_M^{-1} : M \rightarrow {}^L A \otimes \text{Coinv}(M)_L \quad m \mapsto m_{\langle -1 \rangle}^{(1)} \otimes S(m_{\langle -1 \rangle}^{(2)}) \cdot m_{\langle 0 \rangle}.$$

² In the arXiv version of [3], a more restrictive notion of a comodule of a Hopf algebroid is studied, cf. [3, arXiv version, Definition 2.19]. The total algebra A of any Hopf algebroid \mathcal{A} can be regarded as a monoid in the monoidal category of \mathcal{A} -comodules in this more restrictive sense. In this setting, the category of A -modules in the category of \mathcal{A} -comodules, and the category of modules for the base algebra L of \mathcal{A} , were proven to be equivalent, without any further (equalizer preserving) assumption, see [3, Theorem 3.26 and Remark 3.27]. That is, in the arXiv version of [3], a weaker statement is proven under weaker assumptions.

It requires some work to check that $\alpha_M^{-1}(m)$ belongs to ${}^L A \otimes \text{Coinv}(M)_L$. By the assumption that the kernel of (4.4) is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \rightarrow \mathcal{M}_L$, we need to show only that

$$m_{\langle -1 \rangle}^{(1)} \otimes \left(S(m_{\langle -1 \rangle}^{(2)}) \cdot m_{\langle 0 \rangle} \right)_{\langle -1 \rangle} \otimes \left(S(m_{\langle -1 \rangle}^{(2)}) \cdot m_{\langle 0 \rangle} \right)_{\langle 0 \rangle} = m_{\langle -1 \rangle}^{(1)} \otimes 1_A \otimes S(m_{\langle -1 \rangle}^{(2)}) \cdot m_{\langle 0 \rangle}, \quad (4.5)$$

as elements of ${}^L A \otimes A_L \otimes {}_L M_L$, for all $m \in M$. Compose the well defined map

$$A^R \otimes {}^R A_L \otimes {}_L A \rightarrow A^R \otimes {}_R A, \quad a \otimes b \otimes c \mapsto a \otimes S(b)c$$

with the equal maps $(\gamma_R \otimes {}_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R : A \rightarrow A^R \otimes {}^R A_L \otimes {}_L A$ (cf. (2.18)) in order to conclude that, for any $a \in A$,

$$a_{(1)}^{(1)} \otimes S(a_{(1)}^{(2)})a_{(2)} = a^{(1)} \otimes S(a^{(2)}_{(1)})a^{(2)}_{(2)} = a^{(1)} \otimes s_R \circ \pi_R(a^{(2)}) = a \otimes 1_A. \quad (4.6)$$

In (4.6), in the second equality (2.20) was used, and the last equality follows by the counitality of γ_R . Using the left A -linearity of the coaction $\tau : M \rightarrow \mathcal{W} \otimes_A M \cong A_L \otimes {}_L M$, anti-comultiplicativity of the antipode (cf. Proposition 2.3(2)), coassociativity of τ and γ_R and finally (4.6), the left hand side of (4.5) is computed to be equal to

$$\begin{aligned} m_{\langle -2 \rangle}^{(1)} &\otimes S(m_{\langle -2 \rangle}^{(2)})_{(1)} m_{\langle -1 \rangle} \otimes S(m_{\langle -2 \rangle}^{(2)})_{(2)} \cdot m_{\langle 0 \rangle} \\ &= m_{\langle -2 \rangle}^{(1)} \otimes S(m_{\langle -2 \rangle}^{(2)(2)}) m_{\langle -1 \rangle} \otimes S(m_{\langle -2 \rangle}^{(2)(1)}) \cdot m_{\langle 0 \rangle} \\ &= m_{\langle -1 \rangle}^{(1)(1)} \otimes S(m_{\langle -1 \rangle}^{(2)})_{(1)} m_{\langle -1 \rangle} \otimes S(m_{\langle -1 \rangle}^{(1)(2)}) \cdot m_{\langle 0 \rangle} \\ &= m_{\langle -1 \rangle}^{(1)} \otimes 1_A \otimes S(m_{\langle -1 \rangle}^{(2)}) \cdot m_{\langle 0 \rangle}. \end{aligned}$$

Thus it follows that $\alpha_M^{-1}(m)$ belongs to ${}^L A \otimes \text{Coinv}(M)_L$ for all $m \in M$, as stated. By (2.20) and the counitality of τ , $\alpha_M \circ \alpha_M^{-1}(m) = m$, for all $m \in M$. It follows by (4.6) that also $\alpha_M^{-1} \circ \alpha_M(a \otimes m) = a \otimes m$, for all $a \otimes m \in {}^L A \otimes \text{Coinv}(M)_L$.

The coinvariants of the left \mathcal{W} -comodule ${}^L A \otimes N_L$ are the elements of

$$\text{Coinv}({}^L A \otimes N_L) = \left\{ \sum_i a_i \otimes n_i \in {}^L A \otimes N_L \mid \sum_i a_i \otimes n_i = \sum_i s_R \circ \pi_R(a_i) \otimes n_i \right\},$$

hence the map

$$\beta_N : \text{Coinv}({}^L A \otimes N_L) \rightarrow N \quad \sum_i a_i \otimes n_i \mapsto \sum_i n_i \cdot \pi_L \circ S(a_i) \equiv \sum_i n_i \cdot \pi_L(a_i)$$

is a right L -module map and natural in N . It is an isomorphism with inverse

$$\beta_N^{-1} : N \rightarrow \text{Coinv}({}^L A \otimes N_L) \quad n \mapsto 1_A \otimes n.$$

■

An analogous result for right-right Hopf modules over \mathcal{A}_R can be obtained by applying Theorem 4.2 to the Hopf algebroid \mathcal{A}_{cop}^{op} .

Proposition 4.3. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and (M, τ) be a left-left Hopf module over \mathcal{A}_L . Then $\text{Coinv}(M)$ is a k -direct summand of M .*

Proof. In light of (4.6), the canonical inclusion $\text{Coinv}(M) \rightarrow M$ is split by the k -module map

$$E_M : M \rightarrow \text{Coinv}(M) \quad m \mapsto S(m_{\langle -1 \rangle}) \cdot m_{\langle 0 \rangle}. \quad (4.7)$$

■

As the next step towards our goal, let us assume that $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid such that the module A_L is finitely generated and projective. Under this assumption we equip \mathcal{A}^*

with the structure of a left-left Hopf module over \mathcal{A}_L . Similarly, in the case when the module ${}^R A$ is finitely generated and projective, we equip \mathcal{A}^* with the structure of a right-right Hopf module over \mathcal{A}_R .

Let $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ be dual bases for the module A_L . A left \mathcal{A}_L -comodule structure on \mathcal{A}^* can be introduced via the L -action

$${}_L \mathcal{A}^* : \quad l \cdot \phi^* := \phi^* \leftarrow S \circ s_L(l) \quad \text{for } l \in L, \phi^* \in \mathcal{A}^*$$

and the left coaction

$$\tau_L : \mathcal{A}^* \rightarrow A_L \otimes {}_L \mathcal{A}^* \quad \phi^* \mapsto \sum_i b_i \otimes \chi^{-1}(\beta_*^i) \phi^*. \quad (4.8)$$

Similarly, let $\{k_j\} \subset A$ and $\{\kappa^j\} \subset \mathcal{A}$ be dual bases for the module ${}^R A$. A right \mathcal{A}_R -comodule structure on \mathcal{A}^* can be introduced by the right R -action

$$\mathcal{A}^*_R : \quad \phi^* \cdot r := \phi^* \leftarrow s_R(r) \quad \text{for } r \in R, \phi^* \in \mathcal{A}^*$$

and the right coaction

$$\tau_R : \mathcal{A}^* \rightarrow \mathcal{A}^*_R \otimes {}^R A \quad \phi^* \mapsto \sum_i \chi^{-1}(\pi_L \circ t_R \circ {}^* \kappa^i \circ S) \phi^* \otimes k_i, \quad (4.9)$$

where $\chi : \mathcal{A}^* \rightarrow \mathcal{A}_*$ is the algebra anti-isomorphism (2.27). Note that the coactions (4.8) and (4.9) are independent of the choice of the dual bases.

Lemma. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and consider the algebra anti-isomorphism χ in (2.27). For any $a, b \in A$ and $\psi_* \in \mathcal{A}_*$,*

$$\chi(\chi^{-1}(\psi_*) \leftarrow a)(b) = \pi_L \circ t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b)a_{(2)}) \quad \text{and} \quad (4.10)$$

$$\chi(\chi^{-1}(\psi_*) \leftarrow S(a))(b) = \pi_L(a^{(1)}t_L \circ \psi_*(S(a^{(2)}b))). \quad (4.11)$$

Proof. By the form of χ and its inverse, for $a, b \in A$ and $\psi_* \in \mathcal{A}_*$,

$$\chi(\chi^{-1}(\psi_*) \leftarrow a)(b) = \pi_L(b^{(2)}t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b^{(1)}_{(1)})a_{(2)}b^{(1)}_{(2)})).$$

For any $a \in A$ and $\psi_* \in \mathcal{A}_*$, there is a well defined map $A_L \otimes {}_L \mathcal{A}^R \otimes {}^R A \rightarrow A$, $x \otimes y \otimes z \mapsto zt_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}x)a_{(2)}y)$. Composing it with the equal maps $(\gamma_L \otimes {}^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L : A \rightarrow A_L \otimes {}_L \mathcal{A}^R \otimes {}^R A$, we conclude that

$$\chi(\chi^{-1}(\psi_*) \leftarrow a)(b) = \pi_L(b^{(2)}^{(2)}t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b_{(1)})a_{(2)}b_{(2)}^{(1)})).$$

Applying the right bialgebroid analogue of (2.6) (in the first equality), counitality of γ_R (in the second equality), (2.5), (2.17) and the left R -linearity of π_R (in the third equality), and (2.1) together with the counitality of γ_L (in the last equality), we conclude that

$$\begin{aligned} \chi(\chi^{-1}(\psi_*) \leftarrow a)(b) &= \pi_L(b^{(2)}^{(2)}t_R \circ \pi_R(t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b_{(1)})a_{(2)})b_{(2)}^{(1)})) \\ &= \pi_L(t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b_{(1)})a_{(2)})b_{(2)}) \\ &= \pi_L \circ t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b_{(1)})a_{(2)}s_L \circ \pi_L(b_{(2)})) \\ &= \pi_L \circ t_R \circ \pi_R(s_L \circ \psi_*(a_{(1)}b)a_{(2)}). \end{aligned}$$

Hence

$$\chi(\chi^{-1}(\psi_*) \leftarrow S(a))(b) = \pi_L \circ t_R \circ \pi_R(s_L \circ \psi_*(S(a)_{(1)}b)S(a)_{(2)}) = \pi_L(a^{(1)}t_L \circ \psi_*(S(a^{(2)}b))),$$

where the last equality follows by Proposition 2.3 (2). ■

Proposition 4.4. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid.*

1) *Introduce the left A -module*

$${}_A\mathcal{A}^* : \quad a \cdot \phi^* := \phi^* \leftarrow S(a) \quad \text{for } a \in A, \phi^* \in \mathcal{A}^*.$$

If the module A_L is finitely generated and projective, then $({}_A\mathcal{A}^, \tau_L)$ – where τ_L is the map (4.8) – is a left-left Hopf module over \mathcal{A}_L .*

2) *Introduce the right A -module*

$$\mathcal{A}^*_A : \quad \phi^* \cdot a := \phi^* \leftarrow a \quad \text{for } a \in A, \phi^* \in \mathcal{A}^*.$$

*If the module RA is finitely generated and projective, then $(\mathcal{A}^*_A, \tau_R)$ – where τ_R is the map (4.8) – is a right-right Hopf module over \mathcal{A}_R .*

The coinvariants of both Hopf modules $({}_A\mathcal{A}^, \tau_L)$ and $(\mathcal{A}^*_A, \tau_R)$ are the elements of $\mathcal{L}(\mathcal{A}^*)$.*

Proof. 1): We have to show that τ_L is a left A -module map. That is, for all $a \in A$ and $\phi^* \in \mathcal{A}^*$

$$\sum_i b_i \otimes \chi^{-1}(\beta_*^i) (\phi^* \leftarrow S(a)) = \sum_i a_{(1)} b_i \otimes (\chi^{-1}(\beta_*^i) \phi^*) \leftarrow S(a_{(2)}) \quad (4.12)$$

as elements of $A_L \otimes {}_L\mathcal{A}^*$. Note that for all $\phi^*, \psi^* \in \mathcal{A}^*$ and $a \in A$

$$(\phi^* \psi^*) \leftarrow a = (\phi^* \leftarrow a^{(2)}) (\psi^* \leftarrow a^{(1)}). \quad (4.13)$$

Since A_L is finitely generated and projective by assumption, using (4.13) and the dual basis property of $\{b_i\}$ and $\{\beta_*^i\}$, one checks that (4.12) is equivalent to the identity

$$(\chi^{-1}(\psi_*) \leftarrow {}_{s_L} \circ \pi_L(a_{(1)})) (\phi^* \leftarrow S(a_{(2)})) = \sum_i (\chi^{-1}(\beta_*^i) \leftarrow S[{}_{s_L} \circ \psi_*(a_{(1)} b_i) a_{(2)}]) (\phi^* \leftarrow S(a_{(3)}))$$

that is equivalent also to

$$\chi^{-1}(\psi_*) \leftarrow {}_{s_L} \circ \pi_L(a) = \sum_i \chi^{-1}(\beta_*^i) \leftarrow S[{}_{s_L} \circ \psi_*(a_{(1)} b_i) a_{(2)}], \quad (4.14)$$

for all $a \in A$, $\psi_* \in \mathcal{A}_*$. Thus we have to prove (4.14). By (4.11), for any $x \in A$,

$$\begin{aligned} \sum_i \chi(\chi^{-1}(\beta_*^i) \leftarrow S({}_{s_L} \circ \psi_*(a_{(1)} b_i) a_{(2)})) (x) \\ = \sum_i \pi_L \left({}_{s_L} \circ \psi_*(a_{(1)} b_i) a_{(2)}^{(1)} t_L \circ \beta_*^i(S(a_{(2)}^{(2)} x)) \right) \\ = \sum_i \psi_* \left(t_L \circ \pi_L(a_{(2)}^{(1)} t_L \circ \beta_*^i(S(a_{(2)}^{(2)} x)) a_{(1)} b_i) \right). \end{aligned} \quad (4.15)$$

For any $x \in A$ and $\beta_* \in \mathcal{A}_*$, there is a well defined map $A_L \otimes {}_L A^R \otimes {}^R A \rightarrow A$, $a \otimes b \otimes c \mapsto t_L \circ \pi_L(b t_L \circ \beta_*(S(c)x))a$. Composing it with the equal maps $(A_L \otimes \gamma_R) \circ \gamma_L = (\gamma_L \otimes {}^R A) \circ \gamma_R : A \rightarrow A_L \otimes {}_L A^R \otimes {}^R A$, we conclude that (4.15) is equal to

$$\begin{aligned} \sum_i \psi_* \left(t_L \circ \pi_L(a^{(1)} {}_{(2)} t_L \circ \beta_*^i(S(a^{(2)} x)) a^{(1)} {}_{(1)} b_i) \right) &= \sum_i \psi_* \left(a^{(1)} t_L \circ \beta_*^i(S(a^{(2)} x)) b_i \right) \\ &= \psi_* \left(a^{(1)} S(a^{(2)} x) \right) = \psi_* ({}_{s_L} \circ \pi_L(a) x). \end{aligned}$$

That is, $\sum_i \chi^{-1}(\beta_*^i) \leftarrow S({}_{s_L} \circ \psi_*(a_{(1)} b_i) a_{(2)}) = \chi^{-1}(\psi_* ({}_{s_L} \circ \pi_L(a) -))$. Since for any $l \in L$, $\chi^{-1}(\psi_* ({}_{s_L}(l) -)) = \chi^{-1}(\psi_*) \leftarrow {}_{s_L}(l)$, this proves (4.14) hence claim 1).

For the Hopf module $({}_A\mathcal{A}^*, \tau_L)$, a projection onto the coinvariants is given by the map (4.7), that takes the explicit form

$$E_{\mathcal{A}^*} : \mathcal{A}^* \rightarrow \text{Coinv}(\mathcal{A}^*) \quad \phi^* \mapsto \sum_i \chi^{-1}(\beta_*^i) \phi^* \leftarrow S^2(b_i). \quad (4.16)$$

A left s -integral λ^* on \mathcal{A} is a coinvariant, since it is an invariant of the left regular \mathcal{A}^* -module and so for all $a \in A$

$$E_{\mathcal{A}^*}(\lambda^*)(a) = \sum_i \chi^{-1}(\beta_*^i)(1_A) \lambda^*(S^2(b_i)a) = \lambda^*[S^2(t_L \circ \beta_*^i(1_A) b_i) a] = \lambda^*(a).$$

On the other hand, for all $a \in A$

$$\sum_i S(b_i)(a \leftarrow \beta_*^i) = S[t_L \circ \beta_*^i(a_{(1)}) b_i] a_{(2)} = s_R \circ \pi_R(a), \quad (4.17)$$

hence for all $\phi^* \in \mathcal{A}^*$

$$\begin{aligned} E_{\mathcal{A}^*}(\phi^*) \rightarrow a &= \sum_i a^{(2)} t_R \circ \pi_R \left\{ [\phi^* \rightarrow S^2(b_i)a^{(1)}] \leftarrow \beta_*^i \right\} \\ &= \sum_i t_R \circ \pi_R \circ S^2(b_i^{(2)}) a^{(2)} t_R \circ \pi_R \left\{ [\phi^* \rightarrow S^2(b_i^{(1)})a^{(1)}] \leftarrow \beta_*^i \right\} \\ &= \sum_i S(b_{i(2)}) (S^2(b_{i(1)})a)^{(2)} t_R \circ \pi_R \left\{ [\phi^* \rightarrow (S^2(b_{i(1)})a)^{(1)}] \leftarrow \beta_*^i \right\} \\ &= \sum_i S(b_{i(2)}) \{ [\phi^* \rightarrow S^2(b_{i(1)})a] \leftarrow \beta_*^i \} \\ &= \sum_{i,j} S(b_i) \{ [\phi^* \rightarrow S^2(b_j)a] \leftarrow \beta_*^j \beta_*^i \} \\ &= \sum_j s_R \circ \pi_R \{ [\phi^* \rightarrow S^2(b_j)a] \leftarrow \beta_*^j \} = s_R \circ E_{\mathcal{A}^*}(\phi^*)(a). \end{aligned}$$

That is, any coinvariant is an s -integral on \mathcal{A}_R . Here we used (4.16), the right analogue of (2.1), the identity $t_R \circ \pi_R \circ S^2 = S \circ s_R \circ \pi_R$, (2.20), the right analogue of (2.3), the identity $\gamma_R[(\phi^* \rightarrow a) \leftarrow \psi_*] = (\phi^* \rightarrow a^{(1)}) \leftarrow \psi_* \otimes a^{(2)}$, holding true for all $a \in A$, $\phi^* \in \mathcal{A}^*$ and $\psi_* \in \mathcal{A}_*$, the dual basis property and (4.17).

2): The proof is analogous to part 1), so we do not repeat the details. We have to show that τ_R is a right A -module map. That is, for all $a \in A$ and $\phi^* \in \mathcal{A}^*$

$$\sum_j \chi^{-1}(\pi_L \circ t_R \circ {}^* \kappa^j \circ S)(\phi^* \leftarrow a) \otimes k_j = \sum_j (\chi^{-1}(\pi_L \circ t_R \circ {}^* \kappa^j \circ S)\phi^*) \leftarrow a^{(1)} \otimes k_j a^{(2)} \quad (4.18)$$

as elements of $\mathcal{A}^*_R \otimes {}^R A$. By similar steps used to show the equivalence of (4.12) and (4.14), the identity (4.18) is shown to be equivalent to

$$\chi^{-1}(\pi_L \circ t_R \circ {}^* \psi \circ S) \leftarrow t_R \circ \pi_R(a) = \sum_j \chi^{-1}(\pi_L \circ t_R \circ {}^* \kappa^j \circ S) \leftarrow a^{(1)} s_R \circ {}^* \psi(k_j a^{(2)}), \quad (4.19)$$

for all $a \in A$, ${}^* \psi \in {}^* \mathcal{A}$. Verification of (4.19) goes by similar steps used to prove (4.14), making use of (4.10). A projection onto the coinvariants is given by

$$E : \phi^* \mapsto \sum_j \chi^{-1}(\pi_L \circ t_R \circ {}^* \kappa^j \circ S) \phi^* \leftarrow S(k_j).$$

By similar steps in part 1), one checks that $E(\lambda^*) = \lambda^*$, for $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$, and $E(\phi^*) \in \mathcal{L}(\mathcal{A}^*)$, for any $\phi^* \in \mathcal{A}^*$. ■

Note that, if in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ both modules A_L and ${}^R A$ are finitely generated and projective, then the dual bases $\{b_i\} \subset A$, $\{\beta_*^i\} \subset \mathcal{A}_*$ for A_L , and $\{k_j\} \subset A$, $\{*\kappa^j\} \subset {}^*\mathcal{A}$ for ${}^R A$ are related via the identity

$$\sum_i \beta_*^i \otimes S(b_i) = \sum_j \pi_L \circ t_R \circ *\kappa_j \circ S \otimes k_j$$

in $\mathcal{A}_*^R \otimes {}^R A$. In particular, in this case the projections $\mathcal{A}^* \rightarrow \mathcal{L}(\mathcal{A}^*)$ in parts 1) and 2) of Proposition 4.4 coincide and (4.9) has the alternative form $\tau_R(\phi^*) = \sum_i \chi^{-1}(\beta_*^i) \phi^* \otimes S(b_i)$.

Let us apply Theorem 4.2 to the Hopf modules in Proposition 4.4. If in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the module A_L is finitely generated and projective, and the kernel of the map

$${}^L A \otimes \mathcal{A}^{*L} \rightarrow {}^L A \otimes A_L \otimes {}^L \mathcal{A}^{*L}, \quad a \otimes \phi^* \mapsto a \otimes \tau_L(\phi^*) - a \otimes 1_A \otimes \phi^*$$

is equal to ${}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L$, then we conclude that

$$\alpha_L : {}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L \rightarrow \mathcal{A}^* \quad a \otimes \lambda^* \mapsto \lambda^* \leftarrow S(a) \quad (4.20)$$

is an isomorphism of left-left Hopf modules over \mathcal{A}_L . If the module ${}^R A$ is finitely generated and projective, and the kernel of

$${}^R \mathcal{A}^* \otimes A_R \rightarrow {}^R \mathcal{A}^* \otimes {}^R A \otimes A_R, \quad \phi^* \otimes a \mapsto \tau_R(\phi^*) \otimes a - \phi^* \otimes_R 1_A \otimes a$$

is equal to ${}^R \mathcal{L}(\mathcal{A}^*) \otimes A_R$, then we conclude that

$$\alpha_R : {}^R \mathcal{L}(\mathcal{A}^*) \otimes A_R \rightarrow \mathcal{A}^* \quad \lambda^* \otimes a \mapsto \lambda^* \leftarrow a \quad (4.21)$$

is an isomorphism of right-right Hopf modules over \mathcal{A}_R . (The right L -module structure on $\mathcal{L}(\mathcal{A}^*)$ is given by $\lambda^* \cdot l := \lambda^* \leftarrow s_L(l)$ and the left R -module structure is given by $r \cdot \lambda^* := \lambda^* \leftarrow t_R(r)$ – see the explanation after (4.2).)

Corollary 4.5. *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, such that all of the modules A^R , ${}^R A$, ${}^L A$ and A_L are finitely generated and projective, there exist non-zero elements in all of $\mathcal{L}(\mathcal{A}^*)$, $\mathcal{L}({}^*\mathcal{A})$, $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$.*

Proof. Since both modules A^R and A_L are finitely generated and projective by assumption, it follows from Proposition 4.4 and Theorem 4.2 that the map (4.20) is an isomorphism, hence there exist non-zero elements in $\mathcal{L}(\mathcal{A}^*)$.

For any element $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$, $\lambda^* \circ S$ is a (possibly zero) element of $\mathcal{L}({}^*\mathcal{A})$ by Scholium 2.10. Now we claim that it is excluded by the bijectivity of the map (4.20) that $\lambda^* \circ S = 0$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$. For if so, then by the surjectivity of the map (4.20) we have $\phi^*(1_A) = 0$ for all $\phi^* \in \mathcal{A}^*$. But this is impossible, since $\pi_R(1_A) = 1_R$, by definition.

It follows from Scholium 2.10, 3.b) and 4.b) that also $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$ must contain non-zero elements. ■

Since none of the duals of a Hopf algebroid is known to be a Hopf algebroid, it does not follow from Theorem 4.2, however, that for a Hopf algebroid, in which the total algebra is finitely generated and projective as a module over the base algebra, also $\mathcal{L}(A)$ and $\mathcal{R}(A)$ contain non-zero elements. At the moment we do not know under what necessary conditions the existence of non-zero integrals in a Hopf algebroid follows.

It is well known ([28], Proposition 4) that the antipode of a finitely generated and projective Hopf algebra over a commutative ring is bijective. We do not know whether a result of the same strength holds true on Hopf algebroids. Our present understanding of this question is formulated in

Proposition 4.6. *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, such that all of the modules A^R , ${}^R A$, ${}^L A$ and A_L are finitely generated and projective, the following statements are equivalent:*

1) The antipode S is bijective.

2) There exists an invariant $\sum_k x_k \otimes \lambda_k^*$ of the left A -module ${}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ – defined via left multiplication in the first factor – with respect to \mathcal{A}_L , satisfying in addition $\sum_k \lambda_k^*(x_k) = 1_R$. (The right R -module structure of $\mathcal{L}(\mathcal{A}^*)$ is defined by the restriction of the action on $(\mathcal{A}^*)^R$, i.e. as $\lambda^* \cdot r := \lambda^*(\text{ }_R t_R(r))$.)

Proof. For any invariant $\sum_k x_k \otimes \lambda_k^*$ of the left A -module ${}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ and any element $a \in A$, the identities

$$\begin{aligned} \sum_k S(a)x_k^{(1)} \otimes x_k^{(2)} \otimes \lambda_k^* &= \sum_k x_k^{(1)} \otimes ax_k^{(2)} \otimes \lambda_k^* \quad \text{and} \\ \sum_k ax_k^{(1)} \otimes S(x_k^{(2)}) \otimes \lambda_k^* &= \sum_k x_k^{(1)} \otimes S(x_k^{(2)})a \otimes \lambda_k^* \end{aligned}$$

hold true as identities in ${}^R A^R \otimes {}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ and in ${}^R A^R \otimes {}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$, respectively.

2) \Rightarrow 1): In terms of the invariant $\sum_k x_k \otimes \lambda_k^*$, the inverse of the antipode is constructed explicitly as

$$S^{-1} : A \rightarrow A \quad a \mapsto \sum_k (\lambda_k^* \leftarrow a) \rightarrow x_k.$$

1) \Rightarrow 2) If S is bijective then in the case of the Hopf algebroid \mathcal{A}_{cop} the isomorphism (4.20) takes the form

$$\alpha_L^{cop} : A^L \otimes {}^L \mathcal{L}(\mathcal{A}) \rightarrow {}^* \mathcal{A} \quad a \otimes {}^* \lambda \mapsto {}^* \lambda \leftarrow S^{-1}(a),$$

where the left L -module structure on $\mathcal{L}(\mathcal{A})$ is defined by $l \cdot {}^* \lambda := {}^* \lambda \leftarrow t_L(l)$.

In terms of $\sum_k x_k \otimes {}^* \lambda_k := (\alpha_L^{cop})^{-1}(\pi_R)$, the required invariant of ${}^R A \otimes \mathcal{L}(\mathcal{A}^*)^R$ is given by $\sum_k x_k \otimes {}^* \lambda_k \circ S^{-1}$. \blacksquare

In a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, in which all of the modules A_L , ${}_L A$, A^R and ${}^R A$ are finitely generated and projective, the extensions $s_R : R \rightarrow A$ and $t_L : L^{op} \rightarrow A$ satisfy the left depth two (or D2, for short) condition and the extensions $t_R : R^{op} \rightarrow A$ and $s_L : L \rightarrow A$ satisfy the right D2 condition of [19]. If furthermore S is bijective then all the four extensions satisfy both the left and the right D2 conditions. This means ([19], Lemma 3.7) in the case of $s_R : R \rightarrow A$, for example, the existence of finite sets (the so called D2 quasi-bases) $\{d_k\} \subset A^R \otimes {}_R A$, $\{\delta_k\} \subset {}_R \text{End}_R({}_R A^R)$, $\{f_l\} \subset A^R \otimes {}_R A$ and $\{\phi_l\} \subset {}_R \text{End}_R({}_R A^R)$ satisfying

$$\begin{aligned} \sum_k d_k \cdot m_A \circ (\delta_k \otimes {}_R A)(u) &= u \quad \text{and} \\ \sum_l m_A \circ (A^R \otimes \phi_l)(u) \cdot f_l &= u \end{aligned}$$

for all elements u in $A^R \otimes {}_R A$, where the A - A bimodule structure on $A^R \otimes {}_R A$ is defined by left multiplication in the first factor and right multiplication in the second factor.

The D2 quasi-bases for the extension $s_R : R \rightarrow A$ can be constructed in terms of the invariants $\sum_i x_i \otimes \lambda_i^* := \alpha_L^{-1}(\pi_R)$ and $\sum_j x'_j \otimes {}^* \lambda'_j := (\alpha_L^{cop})^{-1}(\pi_R)$ via the requirements that

$$\begin{aligned} \sum_k d_k \otimes \delta_k &= \sum_i x_{i(1)}^{(1)} \otimes S(x_{i(1)}^{(2)}) \otimes [\lambda_i^* \leftarrow S(x_{i(2)})] \rightarrow \text{ } \quad \text{and} \\ \sum_l \phi_l \otimes f_l &= \sum_j \text{ } \leftarrow [x'_{j(1)} \rightarrow \pi_L \circ s_R \circ {}^* \lambda'_j \circ S^{-1}] \otimes x'_{j(2)}^{(1)} \otimes S(x'_{j(2)}^{(2)}) \end{aligned}$$

as elements of $A^R \otimes {}_R A^L \otimes {}_L [{}_R \text{End}_R({}_R A^R)]$ and of $[{}_R \text{End}_R({}_R A^R)]_L \otimes {}_L A^R \otimes {}_R A$, respectively. (The L - L bimodule structure on ${}_R \text{End}_R({}_R A^R)$ is given by

$$l_1 \cdot \Psi \cdot l_2 = s_L(l_1) \Psi(\text{ }_L) s_L(l_2) \quad \text{for } l_1, l_2 \in L, \Psi \in {}_R \text{End}_R({}_R A^R).$$

The D2 property of the extensions $t_R : R^{op} \rightarrow A$, $s_L : L \rightarrow A$ and $t_L : L^{op} \rightarrow A$ follows by applying these formulae to the Hopf algebroids \mathcal{A}_{cop} , \mathcal{A}_{cop}^{op} and \mathcal{A}^{op} , respectively.

The following theorem and corollary, characterizing Frobenius Hopf algebroids $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ – that is, Hopf algebroids such that the extensions, given by the source and target maps of the bialgebroids \mathcal{A}_L and \mathcal{A}_R , are Frobenius extensions –, are the main results of this section.

Recall that, for a homomorphism $s : R \rightarrow A$ of k -algebras, the canonical R - A bimodule ${}_R A_A$ is a 1-cell in the additive bicategory of [k -algebras, bimodules, bimodule maps], possessing a right dual, the bimodule ${}_A A_R$. If A is finitely generated and projective as a left R -module, then ${}_R A_A$ possesses also a left dual, the bimodule ${}_A [{}_R \text{Hom}(A, R)]_R$ defined as

$$a \cdot \phi \cdot r = \phi(-a)r \quad \text{for } r \in R, a \in A, \phi \in {}_R \text{Hom}(A, R).$$

A monomorphism of k -algebras $s : R \rightarrow A$ is called a *Frobenius extension* if the module ${}_R A$ is finitely generated and projective and the left and right duals ${}_A A_R$ and ${}_A [{}_R \text{Hom}(A, R)]_R$ of the bimodule ${}_R A_A$ are isomorphic. Equivalently, if A_R is finitely generated and projective and the left and right duals ${}_R A_A$ and ${}_R [\text{Hom}_R(A, R)]_A$ of the bimodule ${}_A A_R$ are isomorphic. This property holds if and only if there exists a *Frobenius system* $(\psi, \sum_i u_i \otimes v_i)$, where $\psi : A \rightarrow R$ is an R - R bimodule map and $\sum_i u_i \otimes v_i$ is an element of $A \otimes_R A$ such that

$$\sum_i s \circ \psi(a u_i) v_i = a = \sum_i u_i s \circ \psi(v_i a) \quad \text{for all } a \in A.$$

Theorem 4.7. *If in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ all modules A^R , ${}^R A$, A_L and ${}_L A$ are finitely generated and projective, then the following statements are equivalent:*

- 1.a) *The map $s_R : R \rightarrow A$ is a Frobenius extension of k -algebras.*
- 1.b) *The map $t_R : R^{\text{op}} \rightarrow A$ is a Frobenius extension of k -algebras.*
- 1.c) *The map $s_L : L \rightarrow A$ is a Frobenius extension of k -algebras.*
- 1.d) *The map $t_L : L^{\text{op}} \rightarrow A$ is a Frobenius extension of k -algebras.*
- 2.a) *The module $\mathcal{L}(\mathcal{A}^*)^L$, defined by $\lambda^* \cdot l := \lambda^* \leftarrow s_L(l)$, is free of rank 1.*
- 2.b) *S is bijective and the module ${}^L \mathcal{L}(\mathcal{A})$, defined by $l \cdot \lambda := {}^* \lambda \leftarrow t_L(l)$, is free of rank 1.*
- 2.c) *The module ${}_R \mathcal{R}({}_* \mathcal{A})$, defined by $r \cdot {}^* \rho := s_R(r) \rightarrow {}^* \rho$, is free of rank 1.*
- 2.d) *S is bijective and the module $\mathcal{R}(\mathcal{A}_*)_R$, defined by $\rho_* \cdot r := t_R(r) \rightarrow \rho_*$, is free of rank 1.*
- 3.a) *There exists an element $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ such that the map*

$$\mathcal{F} : A \rightarrow \mathcal{A}^* \quad a \mapsto \lambda^* \leftarrow a \tag{4.22}$$

is bijective.

3.b) *S is bijective and there exists an element ${}^* \lambda \in \mathcal{L}({}_* \mathcal{A})$ such that the map $A \rightarrow {}^* \mathcal{A}$, $a \mapsto {}^* \lambda \leftarrow a$ is bijective.*

3.c) *There exists an element ${}^* \rho \in \mathcal{R}({}_* \mathcal{A})$ such that the map $A \rightarrow {}^* \mathcal{A}$, $a \mapsto a \rightarrow {}^* \rho$ is bijective.*

3.d) *S is bijective and there exists an element $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ such that the map $A \rightarrow \mathcal{A}_*$, $a \mapsto a \rightarrow \rho_*$ is bijective.*

4.a) *There exists a left integral $\ell \in \mathcal{L}(\mathcal{A})$ such that the map*

$$\mathcal{F}^* : \mathcal{A}^* \rightarrow A \quad \phi^* \mapsto \phi^* \rightarrow \ell \tag{4.23}$$

is bijective.

4.b) *S is bijective and there exists a left integral $\ell \in \mathcal{L}(\mathcal{A})$ such that the map*

$${}^* \mathcal{F} : {}^* \mathcal{A} \rightarrow A \quad {}^* \phi \mapsto {}^* \phi \rightarrow \ell \tag{4.24}$$

is bijective.

4.c) *There exists a right integral $\wp \in \mathcal{R}(\mathcal{A})$ such that the map ${}_* \mathcal{A} \rightarrow A$, ${}_* \phi \mapsto \wp \leftarrow {}^* \phi$ is bijective.*

4.d) *S is bijective and there exists a right integral $\wp \in \mathcal{R}(\mathcal{A})$ such that the map $\mathcal{A}_* \rightarrow A$, $\phi_* \mapsto \wp \leftarrow \phi_*$ is bijective.*

In particular, the integrals $\lambda^*, {}^*\lambda, {}^*\rho$ and ρ_* on \mathcal{A} satisfying the condition in 3.a), 3.b) 3.c) and 3.d), respectively, are Frobenius functionals themselves for the extensions $s_R : R \rightarrow A$, $t_R : R^{op} \rightarrow A$, $s_L : L \rightarrow A$ and $t_L : L^{op} \rightarrow A$, respectively.

What is more, under the equivalent conditions of the theorem the left integrals $\ell \in \mathcal{L}(\mathcal{A})$ satisfying the conditions in 4.a) and 4.b) can be chosen to be equal, that is, to be a non-degenerate left integral in \mathcal{A} . Similarly, the right integrals $\wp \in \mathcal{R}(\mathcal{A})$ satisfying the conditions in 4.c) and 4.d) can be chosen to be equal, that is to be a non-degenerate right integral in \mathcal{A} .

Proof. 4.a) \Rightarrow 1.a): In terms of the left integral ℓ in 4.a) define $\lambda^* := \mathcal{F}^{*-1}(1_A) \in \mathcal{A}^*$. The element $\ell \otimes \lambda^* \in {}^R\mathcal{L}(A) \otimes \mathcal{L}(\mathcal{A}^*)^R$ is an invariant of the left A -module ${}^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$, hence by Proposition 4.6 the antipode is bijective. Since for all $\phi^* \in \mathcal{A}^*$

$$\phi^* \lambda^* = \mathcal{F}^{*-1}(\phi^* \rightharpoonup 1_A) = \mathcal{F}^{*-1}(s^* \circ \phi^*(1_A) \rightharpoonup 1_A) = s^* \circ \pi^*(\phi^*) \lambda^*,$$

λ^* is an s -integral on \mathcal{A}_R , so in particular an R - R bimodule map ${}^RA^R \rightarrow R$.

Since for all $a \in A$

$$\ell^{(2)} \quad t_R \circ \lambda^* \left(S(a) \ell^{(1)} \right) = a,$$

we have $\mathcal{F}^{*-1}(a) = \lambda^* \leftarrow S(a)$. Hence for all $\phi^* \in \mathcal{A}^*$ and $a \in A$,

$$\phi^*(a) = (\mathcal{F}^{*-1} \circ \mathcal{F}^*)(\phi^*)(a) = \lambda^*(s_R \circ \phi^*(\ell^{(1)}) S(\ell^{(2)}) a) = \phi^*(a \ell^{(1)} s_R \circ \lambda^* \circ S(\ell^{(2)})).$$

Since A^R is finitely generated and projective by assumption, this proves that $\ell^{(1)} s_R \circ \lambda^* \circ S(\ell^{(2)}) = 1_A$. A Frobenius system for the extension $s_R : R \rightarrow A$ is provided by $(\lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$.

1.a) \Rightarrow 2.a): In terms of a Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ for the extension $s_R : R \rightarrow A$, one constructs an isomorphism of right L -modules as

$$\kappa : \mathcal{L}(\mathcal{A}^*) \rightarrow L \quad \lambda^* \mapsto \pi_L \left[\sum_i s_R \circ \lambda^*(u_i) v_i \right] \quad (4.25)$$

$$\text{with inverse} \quad \kappa^{-1} : L \rightarrow \mathcal{L}(\mathcal{A}^*) \quad l \mapsto E_{A^*}(\psi \leftarrow s_L(l)), \quad (4.26)$$

where E_{A^*} is the map (4.16). The right L -linearity of κ follows from the property of the Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ that $\sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a$ for all $a \in A$, the bialgebroid axiom (2.5), and left R -linearity of the map $\lambda^* : {}^RA \rightarrow R$ and the right L -linearity of $\pi_L : {}_LA \rightarrow L$.

The maps κ and κ^{-1} are mutual inverses as

$$\begin{aligned} \kappa^{-1} \circ \kappa(\lambda^*) &= \sum_{i,j} [\chi^{-1}(\beta_*^j) \psi] \leftarrow s_L \circ \pi_L (s_R \circ \lambda^*(u_i) v_i) S^2(b_j) \\ &= \sum_{i,j} [\chi^{-1}(\beta_*^j) \psi] \leftarrow S^2(b_j^{(2)}) t_R \circ \pi_R \left[t_R \circ \pi_R \circ S (s_R \circ \lambda^*(u_i) v_i) S^2(b_j^{(1)}) \right] \\ &= \sum_{i,j} [\chi^{-1}(\beta_*^j) \psi] \leftarrow S^2(b_j^{(2)}) s_L \circ \pi_L \left[S(b_j^{(1)}) s_R \circ \lambda^*(u_i) v_i \right] = \lambda^*, \end{aligned} \quad (4.27)$$

where in the first step we used (4.13), in the second step the fact that by Proposition 2.3 we have $s_L \circ \pi_L = t_R \circ \pi_R \circ S$, then the right analogue of (2.5) and finally in the last step the identity in ${}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R$:

$$\sum_{i,j} [\chi^{-1}(\beta_*^j) \psi] \leftarrow S^2(b_j^{(2)}) \otimes S(b_j^{(1)}) s_R \circ \lambda^*(u_i) v_i = \alpha_R^{-1} \left(\sum_i \psi \leftarrow s_R \circ \lambda^*(u_i) v_i \right) = \lambda^* \otimes 1_A,$$

which follows from the explicit form of the inverse of the map (4.21). In a similar way, also

$$\begin{aligned}
\kappa \circ \kappa^{-1}(l) &= \sum_{i,j} \pi_L [s_R \circ (\chi^{-1}(\beta_*^j) \psi) (s_L(l) S^2(b_j) u_i) v_i] \\
&= \sum_{i,j} \pi_L [s_R \circ (\chi^{-1}(\beta_*^j) \psi) (s_L(l) u_i) v_i S^2(b_j)] \\
&= \sum_{i,j} \pi_L [s_R \circ (\chi^{-1}(\beta_*^j) \psi) (s_L(l) u_i) v_i t_L \circ \pi_L \circ S^2(b_j)] \\
&= \sum_{i,j} \pi_L [s_R \circ (\chi^{-1}(\beta_*^j) \psi) (s_L(l) t_L \circ \pi_L \circ S^2(b_j) u_i) v_i] \\
&= \sum_{i,j} \pi_L \{s_R \circ [(\chi^{-1}(\beta_*^j) \leftarrow t_L \circ \pi_L \circ S^2(b_j)) \psi] (s_L(l) u_i) v_i\} = l,
\end{aligned}$$

where in the last step we used that $\sum_j \chi^{-1}(\beta_*^j) \leftarrow t_L \circ \pi_L \circ S^2(b_j) = \chi^{-1} \left(\sum_j \beta_*^j t_* \circ \pi_L(b_j) \right) = \pi_R$.

2.a) \Rightarrow 3.a): If $\kappa : \mathcal{L}(\mathcal{A}^*)^L \rightarrow L$ is an isomorphism of L -modules then $\pi_R \circ s_L \circ \kappa : {}^R\mathcal{L}(\mathcal{A}^*) \rightarrow R$ is an isomorphism of R -modules. Introduce the cyclic and separating generator $\lambda^* := \kappa^{-1}(1_L)$ for the module $\mathcal{L}(\mathcal{A}^*)^L$. The map \mathcal{F} in (4.22) is equal to $\alpha_R \circ (\kappa^{-1} \circ \pi_L \circ t_R \otimes A_R)$ – where α_R is the isomorphism (4.21) – hence it is bijective.

3.a) \Rightarrow 4.a), 4.b): A Frobenius system for the extension $s_R : R \rightarrow A$ is given in terms of the dual bases $\{b_i\} \subset A$ and $\{\beta_i^*\} \subset \mathcal{A}^*$ for the module A^R as $(\lambda^*, \sum_i b_i \otimes \mathcal{F}^{-1}(\beta_i^*))$.

The element $\ell := \sum_i b_i t_L \circ \pi_L \circ \mathcal{F}^{-1}(\beta_i^*)$ is a left integral in \mathcal{A} . Using the identities

$$\begin{aligned}
\lambda^* \rightharpoonup \ell &= s_R \circ \lambda^* \left[\sum_i b_i t_L \circ \pi_L \circ \mathcal{F}^{-1}(\beta_i^*) \right] = t_L \circ \pi_L \left[\sum_i s_R \circ \lambda^*(b_i) \mathcal{F}^{-1}(\beta_i^*) \right] = 1_A, \\
\ell^{(1)} \otimes S(\ell^{(2)}) &= \sum_i b_i s_R \circ \lambda^* \left[\mathcal{F}^{-1}(\beta_i^*) \ell^{(1)} \right] \otimes S(\ell^{(2)}) = \sum_i b_i \otimes S \left[\ell^{(2)} t_R \circ \lambda^*(\ell^{(1)}) \right] \mathcal{F}^{-1}(\beta_i^*) \\
&= \sum_i b_i \otimes \mathcal{F}^{-1}(\beta_i^*),
\end{aligned}$$

one checks that the inverse of the map \mathcal{F}^* in (4.23) is given by $\mathcal{F} \circ S$. This implies, in particular, that S is bijective.

The inverse of the map ${}^*\mathcal{F}$ in (4.24) – defined in terms of the same left integral ℓ – is the map

$$A \rightarrow {}^*\mathcal{A} \quad a \mapsto \lambda^* \circ S \leftarrow S^{-1}(a).$$

1.a) \Leftrightarrow 1.d): The datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R : R \rightarrow A$ if and only if $(\pi_L \circ s_R \circ \psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for $t_L : L^{op} \rightarrow A$, where $\pi_L \circ s_R : R \rightarrow L^{op}$ was claimed to be an isomorphism of k -algebras in part 1) of Proposition 2.3.

1.a) \Rightarrow 1.c): We have already seen that $1.a) \Rightarrow 3.a) \Rightarrow S$ is bijective. If the datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R : R \rightarrow A$ then $(\pi_L \circ s_R \circ \psi \circ S^{-1}, S(v_i) \otimes S(u_i))$ is a Frobenius system for $s_L : L \rightarrow A$.

4.c) \Rightarrow 1.c) \Rightarrow 2.c) \Rightarrow 3.c) \Rightarrow 4.c), 1.c) \Leftrightarrow 1.b) and 1.c) \Rightarrow 1.a) \Rightarrow 2.a) \Rightarrow 3.a) \Rightarrow 4.a), 1.a) \Leftrightarrow 1.d) and 1.a) \Rightarrow 1.c) to the Hopf algebroid \mathcal{A}_{cop}^{op} .

1.b) \Rightarrow 2.b) \Rightarrow 3.b) \Rightarrow 4.b) \Rightarrow 1.b): We have seen that $1.b) \Leftrightarrow 1.c) \Rightarrow S$ is bijective. Hence we can apply 1.a) \Rightarrow 2.a) \Rightarrow 3.a) \Rightarrow 4.a) \Rightarrow 1.a) to the Hopf algebroid \mathcal{A}_{cop} .

1.d) \Rightarrow 2.d) \Rightarrow 3.d) \Rightarrow 4.d) \Rightarrow 1.d) follows by applying 1.b) \Rightarrow 2.b) \Rightarrow 3.b) \Rightarrow 4.b) \Rightarrow 1.b) to the Hopf algebroid \mathcal{A}_{cop}^{op} . \blacksquare

Based on Theorem 4.7, we obtain the following generalization of Theorem 4.1.

Corollary 4.8. *For any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, the following assertions are equivalent.*

1.a) Both maps $s_R : R \rightarrow A$ and $t_R : R^{op} \rightarrow A$ are Frobenius extensions of k -algebras.

- 1.b) Both maps $s_L : L \rightarrow A$ and $t_L : L^{op} \rightarrow A$ are Frobenius extensions of k -algebras.
- 2.a) The module A^R is finitely generated and projective and there exists an element $\lambda^* \in \mathcal{L}(A^*)$ such that the map $\mathcal{F} : A \rightarrow A^*$, $a \mapsto \lambda^* \leftarrow a$ is bijective.
- 2.b) S is bijective, the module ${}^R A$ is finitely generated and projective and there exists an element ${}^* \lambda \in \mathcal{L}({}^* A)$ such that the map $A \rightarrow {}^* A$, $a \mapsto {}^* \lambda \leftarrow a$ is bijective.
- 2.c) The module ${}_L A$ is finitely generated and projective and there exists an element ${}_* \rho \in \mathcal{R}({}_* A)$ such that the map $A \rightarrow {}_* A$, $a \mapsto a \rightarrow {}_* \rho$ is bijective.
- 2.d) S is bijective, the module A_L is finitely generated and projective and there exists an element $\rho_* \in \mathcal{R}(A_*)$ such that the map $A \rightarrow A_*$, $a \mapsto a \rightarrow \rho_*$ is bijective.
- 3.a) There exists a non-degenerate left integral, that is an element $\ell \in \mathcal{L}(A)$ such that both maps $\mathcal{F}^* : A^* \rightarrow A$, $\phi^* \mapsto \phi^* \rightarrow \ell$ and ${}^* \mathcal{F} : {}^* A \rightarrow A$, ${}^* \phi \mapsto {}^* \phi \rightarrow \ell$ are bijective.
- 3.b) There exists a non-degenerate right integral, that is, an element $\wp \in \mathcal{R}(A)$ such that both maps ${}_* A \rightarrow A$, ${}_* \phi \mapsto \wp \leftarrow {}_* \phi$ and $A_* \rightarrow A$, $\phi_* \mapsto \wp \leftarrow \phi_*$ are bijective.

Proof. 1.a) \Leftrightarrow 1.b): This follows by the same reasoning used to prove 1.a) \Leftrightarrow 1.d) and 1.b) \Leftrightarrow 1.c) in Theorem 4.7.

1.a) \Rightarrow 2.a): Since $s_R : R \rightarrow A$ is a Frobenius extension by assumption, the modules A^R and ${}_R A$ (hence also A_L) are finitely generated and projective by definition. Similarly, since $t_R : R^{op} \rightarrow A$ is a Frobenius extension, the modules ${}^R A$ and ${}_L A$ are finitely generated and projective. Thus this implication follows by Theorem 4.7 1.a) \Rightarrow 3.a).

2.a) \Rightarrow 3.a) and S is bijective: This is proven by repeating the same steps used to prove 3.a) \Rightarrow 4.a) and 4.b) in Theorem 4.7.

3.a) \Rightarrow 1.a) and S is bijective: Putting $\lambda^* := \mathcal{F}^{*-1}(1_A)$, the map $a \mapsto (\lambda^* \leftarrow a) \rightarrow \ell$ is checked to be the inverse of S .

For any $r \in R$, $\mathcal{F}^*(r\lambda^*(-)) = t_R(r) = \mathcal{F}^*(\lambda^* \leftarrow s_R(r))$. So by the bijectivity of \mathcal{F}^* , we conclude that λ^* is a left R -module map ${}_R A \rightarrow R$. Therefore the module ${}_R A$ is finitely generated and projective with dual basis $\lambda^*(S(-)\ell^{(1)}) \otimes \ell^{(2)} \in {}^* A_R \otimes {}^R A$. The module A_L is finitely generated and projective by Lemma 2.6. Applying the same reasoning to the Hopf algebroid \mathcal{A}_{cop} , we conclude that by the bijectivity of ${}^* \mathcal{F}$ also the modules A^R and ${}_L A$ are finitely generated and projective. Hence the claim follows by Theorem 4.7, 4.a) \Rightarrow 1.a) and 1.b).

1.b) \Leftrightarrow 2.c) \Leftrightarrow 3.b): This follows by applying 1.a) \Leftrightarrow 2.a) \Leftrightarrow 3.a) to the Hopf algebroid \mathcal{A}_{cop}^{op} .

1.a) \Leftrightarrow 2.b): Since we proved that 1.a) implies the bijectivity of the antipode, we can apply 1.a) \Leftrightarrow 2.a) to the Hopf algebroid \mathcal{A}_{cop} .

1.b) \Leftrightarrow 2.d): This follows by applying 1.a) \Leftrightarrow 2.b) to the Hopf algebroid \mathcal{A}_{cop}^{op} . ■

It is proven in ([6], Theorem 5.17) that under the equivalent conditions of Theorem 4.7 the duals, A^* , ${}^* A$, ${}_* A$ and A_* of the Hopf algebroid \mathcal{A} , possess (anti-) isomorphic Hopf algebroid structures.

The Hopf algebroids, satisfying the equivalent conditions of Theorem 4.7, provide examples of distributive Frobenius double algebras [38]. (Notice that the integrals, which we call non-degenerate, are called Frobenius integrals in [38]).

Our result naturally raises the question, under what conditions on the base algebra the equivalent conditions of Theorem 4.7 hold true. That is, what is the generalization of Pareigis' condition – the triviality of the Picard group of the commutative base ring of a Hopf algebra – to the non-commutative base algebra of a Hopf algebroid. We are going to return to this problem in a different publication.

5 The Quasi-Frobenius property

It is known ([28], Theorem added in proof), that any finitely generated projective Hopf algebra over a commutative ring k is (both a left and a right) quasi-Frobenius extension of k in the sense of [24]. In this section we examine in what Hopf algebroids is the total algebra (a left or a right) quasi-Frobenius extension of the base algebra.

The quasi-Frobenius property of an extension $s : R \rightarrow A$ of k -algebras has been introduced by Müller [24] as a weakening of the Frobenius property (see the paragraph preceeding Theorem 4.7). The extension $s : R \rightarrow A$ is *left quasi-Frobenius* (or left QF, for short) if the module ${}_R A$ is finitely generated and projective (hence the bimodule ${}_R A_A$ possesses both a right dual ${}_A A_R$ and a left dual ${}_A [{}_R \text{Hom}(A, R)]_R$) and the bimodule ${}_A A_R$ is a direct summand in a finite direct sum of copies of ${}_A [{}_R \text{Hom}(A, R)]_R$.

The extension $s : R \rightarrow A$ is *right QF* if s , considered as a map $R^{op} \rightarrow A^{op}$, is a left QF extension. That is, if the module A_R is finitely generated and projective and the left dual bimodule ${}_R A_A$ is a direct summand in a finite direct sum of copies of the right dual bimodule ${}_R [\text{Hom}_R(A, R)]_A$.

To our knowledge it is not known whether the notions of left and right QF extensions are equivalent (except in particular cases, such as central extensions, where the answer turns out to be affirmative [30]; and Frobenius extensions, which are also both left and right QF [24]).

A powerful characterization of a Frobenius extension $s : R \rightarrow A$ is the existence of a Frobenius system – see the paragraph preceeding Theorem 4.7. In the following lemma a generalization to quasi-Frobenius extensions is introduced:

Lemma 5.1. *1) An algebra extension $s : R \rightarrow A$ is left QF if and only if the module ${}_R A$ is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R \text{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \otimes_R A$ satisfying*

$$\begin{aligned} \sum_{i,k} u_i^k s \circ \psi_k(v_i^k) &= 1_A \quad \text{and} \\ \sum_i a u_i^k \otimes v_i^k &= \sum_i u_i^k \otimes v_i^k a \quad \text{for all values of } k \text{ and } a \in A. \end{aligned}$$

The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a left QF-system for the extension $s : R \rightarrow A$.

2) An algebra extension $s : R \rightarrow A$ is right QF if and only if the module A_R is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R \text{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \otimes_R A$ satisfying

$$\begin{aligned} \sum_{i,k} s \circ \psi_k(u_i^k) v_i^k &= 1_A \quad \text{and} \\ \sum_i a u_i^k \otimes v_i^k &= \sum_i u_i^k \otimes v_i^k a \quad \text{for all values of } k \text{ and } a \in A. \end{aligned}$$

The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a right QF-system for the extension $s : R \rightarrow A$.

Proof. Let us spell out the proof in case 1): Suppose that there exists a left QF system $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s : R \rightarrow A$. The bimodule ${}_A A_R$ is a direct summand in a finite direct sum of copies of ${}_A [{}_R \text{Hom}(A, R)]_R$ by the existence of A - R bimodule maps

$$\begin{aligned} \Phi_k : {}_R \text{Hom}(A, R) &\rightarrow A & \phi &\mapsto \sum_i u_i^k s \circ \phi(v_i^k) & \text{and} \\ \Phi'_k : A &\rightarrow {}_R \text{Hom}(A, R) & a &\mapsto \psi_k(_ a) \end{aligned}$$

satisfying $\sum_k \Phi_k \circ \Phi'_k = A$.

Conversely, in terms of the A - R bimodule maps $\{\Phi_k : {}_R \text{Hom}(A, R) \rightarrow A\}$ and $\{\Phi'_k : A \rightarrow {}_R \text{Hom}(A, R)\}$, satisfying $\sum_k \Phi_k \circ \Phi'_k = A$, and the dual bases, $\{b_j\} \subset A$ and $\{\beta_j\} \subset {}_R \text{Hom}(A, R)$ for the module ${}_R A$, a left QF system can be constructed as

$$\begin{aligned} \psi_k &:= \Phi'_k(1_A) \in {}_R \text{Hom}_R(A, R) & \text{and} \\ \sum_i u_i^k \otimes v_i^k &:= \sum_j \Phi_k(\beta_j) \otimes b_j \in A \otimes_R A. \end{aligned} \quad \blacksquare$$

Lemma 5.1 implies, in particular, that for a left/right QF extension $R \rightarrow A$, A is finitely generated and projective also as a right/left R -module.

Theorem 5.2. *If in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ all modules A^R , ${}^R A$, A_L and ${}^L A$ are finitely generated and projective, then the following are equivalent:*

- 1.a) $s_R : R \rightarrow A$ is a left QF extension.
- 1.b) $t_L : L^{op} \rightarrow A$ is a left QF extension.
- 1.c) The module $\mathcal{L}(\mathcal{A}^*)^L$ – defined by $\lambda^* \cdot l := \lambda^* \leftarrow s_L(l)$ – is finitely generated and projective.
- 1.d) The module $\mathcal{L}(\mathcal{A}^*)^L$ is flat.
- 1.e) The invariants of the left A -module ${}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L$ – defined via left multiplication in the first factor – with respect to \mathcal{A}_L are the elements of ${}^L \mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{A}^*)^L$.
- 1.f) There exist finite sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ satisfying $\sum_k \lambda_k^* \circ S(\ell_k) = 1_R$.
- 1.g) The left A -module ${}_A \mathcal{A}^*$ – defined by $a \cdot \phi^* := \phi^* \leftarrow S(a)$ – is finitely generated and projective with generator set $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$.

The following properties of \mathcal{A} are also equivalent:

- 2.a) $s_L : L \rightarrow A$ is a right QF extension.
- 2.b) $t_R : R^{op} \rightarrow A$ is a right QF extension.
- 2.c) The module ${}_R \mathcal{R}({}_* \mathcal{A})$ – defined by $r \cdot {}_* \rho := s_R(r) \rightarrow {}_* \rho$ – is finitely generated and projective.
- 2.d) The module ${}_R \mathcal{R}({}_* \mathcal{A})$ is flat.
- 2.e) The invariants of the right A -module ${}_R \mathcal{R}({}_* \mathcal{A}) \otimes A_R$ – defined via right multiplication in the second factor – with respect to \mathcal{A}_R are the elements of ${}_R \mathcal{R}({}_* \mathcal{A}) \otimes \mathcal{R}(\mathcal{A})_R$.
- 2.f) There exist finite sets $\{\wp_k\} \subset \mathcal{R}(\mathcal{A})$ and $\{{}_* \rho_k\} \subset \mathcal{R}({}_* \mathcal{A})$ satisfying $\sum_k {}_* \rho_k \circ S(\wp_k) = 1_L$.
- 2.g) The right A -module ${}_A \mathcal{A}_*$ – defined by ${}_* \phi \cdot a := S(a) \rightarrow {}_* \phi$ – is finitely generated and projective with generator set $\{{}_* \rho_k\} \subset \mathcal{R}({}_* \mathcal{A})$.

If furthermore the antipode is bijective, then conditions 1.a)-1.g) and 2.a)-2.g) are equivalent to each other and also to

- 1.h) The left ${}_A \mathcal{A}$ -module on A – defined by ${}_* \phi \cdot a := {}_* \phi \rightarrow a$ – is finitely generated and projective with generator set $\{\ell_k\} \in \mathcal{L}(\mathcal{A})$.
- 2.h) The right \mathcal{A}_* -module on A – defined by $a \cdot \phi_* := a \leftarrow \phi_*$ – is finitely generated and projective with generator set $\{\wp_k\} \in \mathcal{R}(\mathcal{A})$.

Proof. 1.a) \Leftrightarrow 1.b): The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for the extension $s_R : R \rightarrow A$ if and only if $\{\pi_L \circ s_R \circ \psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for $t_L : L^{op} \rightarrow A$.

1.a) \Rightarrow 1.c): In terms of the left QF system, $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s_R : R \rightarrow A$, the dual bases for the module $\mathcal{L}(\mathcal{A}^*)^L$ are given with the help of the map (4.16) as $\{E_{\mathcal{A}^*}(\psi_k)\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\kappa_k := \pi_L [\sum_i s_R \circ (u_i^k \otimes v_i^k)]\} \subset \text{Hom}_L(\mathcal{L}(\mathcal{A}^*)^L, L)$.

The right L -linearity of the maps $\kappa_k : \mathcal{L}(\mathcal{A}^*) \rightarrow L$ is checked similarly to the right L -linearity of the map (4.25). Notice that for any R - R bimodule map $\psi : {}_R A^R \rightarrow R$ we have

$$\begin{aligned}
E_{\mathcal{A}^*}(\psi \leftarrow s_L(l)) &= \sum_j [\chi^{-1}(\beta_*^j) \psi] \leftarrow s_L(l) S^2(b_j) \\
&= \sum_j [\chi^{-1}(t_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l) \beta_*^j) \psi] \leftarrow S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j) t^* \circ \pi_R \circ t_L(l) \psi] \leftarrow S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j) s^* \circ \pi_R \circ t_L(l) \psi] \leftarrow S^2(b_j) \\
&= \sum_j [\chi^{-1}(s_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l) \beta_*^j) \psi] \leftarrow S^2(b_j) \\
&= E_{\mathcal{A}^*}(\psi) \leftarrow s_L(l),
\end{aligned}$$

for all $l \in L$, where in the first step we used (4.16) and (4.13), in the second step the property of the dual bases $\{b_j\} \subset A$ and $\{\beta_*^j\} \subset \mathcal{A}_*$ that $\sum_j \beta_*^j \otimes s_L(l) b_j = \sum_j t_*(l) \beta_*^j \otimes b_j$ for all $l \in L$ as elements of ${}^L \mathcal{A}_* \otimes A_L$, in the third step the identity $\chi^{-1} \circ t_* = t^* \circ \pi_R \circ s_L$, in the fourth step the fact that by the left R -linearity of ψ we have $t^*(r) \psi = s^*(r) \psi$ for all $r \in R$, in the fifth step

$\chi^{-1} \circ s_* = s^* \circ \pi_R \circ s_L$, and finally $\sum_j \beta_*^j \otimes b_j s_L(l) = \sum_j s_*(l) \beta_*^j \otimes b_j$, holding true for all $l \in L$ as an identity in ${}^L\mathcal{A}_* \otimes A_L$.

The dual basis property of the sets $\{E_{\mathcal{A}^*}(\psi_k)\}$ and $\{\kappa_k\}$ is verified by the property that $\sum_{i,k} E_{\mathcal{A}^*}(\psi_k \leftarrow s_L \circ \kappa_k(\lambda^*)) = \lambda^*$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$, which is checked similarly to (4.27).

1.c) \Rightarrow 1.d) is a standard result.

1.d) \Rightarrow 1.e): Since the module A_L is finitely generated and projective by assumption, the invariants of any left A -module M with respect to \mathcal{A}_L are the elements of the kernel of the map

$$\zeta_M : M \rightarrow {}^L\mathcal{A}_* \otimes M_L \quad m \mapsto \left(\sum_i \beta_*^i \otimes b_i \cdot m \right) - \pi_L \otimes m,$$

where the right L module M_L is defined via t_L , and the sets $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ are dual bases for the module A_L .

The map ζ_A , corresponding to the left regular A -module, is a left L -module map ${}^L A \rightarrow {}^L\mathcal{A}^* \otimes {}^L A_L$ and $\zeta_{A \otimes \mathcal{L}(\mathcal{A}^*)^L} = \zeta_A \otimes \mathcal{L}(\mathcal{A}^*)^L$. Since tensoring with $\mathcal{L}(\mathcal{A}^*)^L$ is an exact functor by assumption, it preserves the kernels, that is the invariants in this case.

1.e) \Rightarrow 1.f): With the help of the map (4.20) introduce

$$\sum_k \ell_k \otimes \lambda_k^* := \alpha_L^{-1}(\pi_R) \in \text{Inv}({}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L) \equiv {}^L \mathcal{L}(A) \otimes \mathcal{L}(\mathcal{A}^*)^L.$$

It satisfies $\sum_k \lambda_k^* \circ S(\ell_k) = \alpha_L \circ \alpha_L^{-1}(\pi_R)(1_A) = 1_R$.

1.f) \Rightarrow 1.a): In terms of the sets $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$, a left QF system for the extension $s_R : R \rightarrow A$ can be constructed as $\{\lambda_k^*, \ell_k^{(1)} \otimes S(\ell_k^{(2)})\}$.

1.f) \Rightarrow 1.g): In terms of the sets $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$, the dual bases for the module ${}_A\mathcal{A}^*$ are given by $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{ _ \rightarrow \ell_k \} \subset {}_A\text{Hom}({}_A\mathcal{A}^*, A)$.

1.g) \Rightarrow 1.f): In terms of the dual bases $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\Xi_k\} \subset {}_A\text{Hom}({}_A\mathcal{A}^*, A)$ one defines the required left integrals $\ell_k := \Xi_k(\pi_R)$ in \mathcal{A} .

The equivalence of the conditions 2.a) – 2.g) follows by applying the above results to the Hopf algebroid \mathcal{A}_{cop}^{op} .

Now assume that S is bijective. Then

1.f) \Leftrightarrow 2.f) follows from Scholium 2.10.

1.f) \Rightarrow 1.h): Scholium 2.8, 1.b) and Scholium 2.10, 3.c) can be used to show that in terms of the sets $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$, the dual bases for the left \mathcal{A} -module on A are given by $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\lambda_k^* \circ S \leftarrow S^{-1}(_)\} \subset {}_{\mathcal{A}}\text{Hom}(A, \mathcal{A})$.

1.h) \Rightarrow 1.f): Let $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\chi_k\} \subset {}_{\mathcal{A}}\text{Hom}(A, \mathcal{A})$ be dual bases for the left \mathcal{A} -module A . For any value of the index k , the element $\chi_k(1_A)$ is an invariant of the left regular \mathcal{A} -module. Hence by the finitely generated projectivity of the module ${}^R A$, it is a t -integral on \mathcal{A}_R . By Scholium 2.10 the elements $\lambda_k^* := \chi_k(1_A) \circ S^{-1}$ are s -integrals on \mathcal{A}_R , satisfying

$$\sum_k \lambda_k^* \circ S(\ell_k) = \pi_R[\sum_k \chi_k(1_A) \rightarrow \ell_k] = 1_R.$$

2.f) \Leftrightarrow 2.h) follows by applying 1.f) \Leftrightarrow 1.h) to the Hopf algebroid \mathcal{A}_{cop}^{op} . ■

If the antipode of a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is bijective then the application of Theorem 5.2 to the Hopf algebroid \mathcal{A}^{op} results equivalent conditions under which the extensions $s_R : R \rightarrow A$ and $t_L : L^{op} \rightarrow A$ are right QF, and $s_L : L \rightarrow A$ and $t_R : R^{op} \rightarrow A$ are left QF.

In order to show that – in contrast to Hopf algebras over commutative rings – not any finitely generated projective Hopf algebroid is quasi-Frobenius, let stand here an example (with bijective antipode) such that the total algebra is finitely generated and projective as a module over the base algebra (in all the four senses listed in (2.16)) and the total algebra is neither a left nor a right QF extension of the base algebra.

The example is taken from ([22], Example 3.1) where it is shown that for any algebra B over a commutative ring k the k -algebra $A := B \otimes_k B^{op}$ has a left bialgebroid structure denoted by \mathcal{A}_L , over the base B , with structural maps

$$\begin{array}{ll}
s_L : B \rightarrow A & b \mapsto b \otimes 1_B \\
t_L : B^{op} \rightarrow A & b \mapsto 1_B \otimes b \\
\gamma_L : A \rightarrow A_B \otimes_B A & b_1 \otimes b_2 \mapsto (b_1 \otimes 1_B) \otimes (1_B \otimes b_2) \\
\pi_L : A \rightarrow B & b_1 \otimes b_2 \mapsto b_1 b_2.
\end{array} \tag{5.1}$$

The bialgebroid \mathcal{A}_L satisfies the Hopf algebroid axioms of [22] with the involutive antipode S , equal to the flip map

$$S : B \otimes_k B^{op} \rightarrow B^{op} \otimes_k B \quad b_1 \otimes b_2 \mapsto b_2 \otimes b_1. \tag{5.2}$$

The reader may check that A has a Hopf algebroid structure also in the sense of this paper with left bialgebroid structure (5.1), antipode (5.2) and right bialgebroid structure $\mathcal{A}_R = (A, B^{op}, S \circ s_L, S \circ t_L, (S \otimes S) \circ \gamma_L^{op} \circ S, \pi_L \circ S)$.

If B is finitely generated and projective as a k -module, then all modules $A^{B^{op}}$, $B^{op} A$, A_B and ${}_B A$ are finitely generated and projective. What is more, we have

Lemma 5.3. *Let B be an algebra over a commutative ring k with trivial center. The following statements are equivalent:*

- 1) *The extension $k \rightarrow B$ is left QF.*
- 2) *The extension $k \rightarrow B$ is right QF.*
- 3) *The extension $B \rightarrow B \otimes_k B^{op}$, $b \mapsto b \otimes 1_B$ is left QF.*
- 4) *The extension $B \rightarrow B \otimes_k B^{op}$, $b \mapsto b \otimes 1_B$ is right QF.*

The equivalence $1) \Leftrightarrow 2)$ is proven in [30] and the rest can be proven using the technics of quasi-Frobenius systems.

In view of Lemma 5.3, it is easy to construct a finitely generated projective Hopf algebroid which is not QF. Let us choose, for example, B to be the algebra of $n \times n$ upper triangle matrices with entries in a commutative ring k . Then B has a trivial center and it is neither a left nor a right QF extension of k . Hence $A = B \otimes_k B^{op}$ is neither a left nor a right QF extension of B .

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